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The Maslov triple index on the Shilov boundary of a classical domain

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Abstract

Let $\mathcal{D}$ be an irreducible Hermitian symmetric space of tube-type, S its Shilov boundary, G its group of holomorphic diffeomorphisms. For a generic triple of points $(\sigma_1, \sigma_2, \sigma_3) \in S \times S \times S$, a characteristic G-invariant $\iota(\sigma_1, \sigma_2, \sigma_3)$, called the *Maslov index* was introduced in [Transform. Groups 6 (2001) 303]. For $\mathcal{D}$ of classical type (i.e. for all cases except for the exceptional domain associated to Albert's algebra), the definition of the Maslov index is extended to all triples, by using a holomorphic embedding of $\mathcal{D}$ into a Siegel disc, which corresponds to an embedding of S into a Lagrangian manifold. When $\mathcal{D}$ is the Lie ball, the extension of the definition is obtained through a realization of S in the Lagrangian manifold of a spinor space.

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1. Introduction

In [4] the present author in collaboration with Ørsted introduced a generalization of the Maslov index[1] for triples of points in the Shilov boundary S of a bounded symmetric domain of tube-type $\mathcal{D}$. However, the Maslov index is defined only for generic triples $(\sigma_1, \sigma_2, \sigma_3) \in S^3$, those such that any two elements of the triple satisfy a condition called transversality.

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[1] The terminology is not quite standard. Some authors call *triple index* or *signature cocyle* what we call *Maslov index* (see [2] for a systematic treatment of the various notions related to the Maslov index).

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Here the definition of the Maslov index is extended to all triples, at least for classical domains. Considering only irreducible domains, there are four infinite series of classical domains, associated to classical (also called special) simple Euclidean Jordan algebras, namely $\mathrm{Sym}(r, \mathbb{R})$, $\mathrm{Herm}(r, \mathbb{C})$, $\mathrm{Herm}(r, \mathbb{H})$ and the *Lorentz algebras* $\mathcal{L}_p$ of rank 2 associated to the light cone in $\mathbb{R}^p$. There is only one simple Euclidean Jordan algebra which is not classical, it is the so-called Albert's algebra, the algebra of 3×3 Hermitian matrices over the octonions $\mathbb{O}$, which will not be considered here. Classical Euclidean Jordan algebras J can be characterized abstractly by the existence of a non-trivial representation (in the sense of Jordan algebras) of J.

The extended Maslov index has all the expected properties. It takes integral values, it is invariant by the group G of holomorphic diffeomorphisms of the domain $\mathcal{D}$ (G may be thought of as the group of conformal transformations of S in the sense of [1]), it is skew-symmetric with respect to permutation of the indices and satisfies a *cocyle relation*.

In Section 2, the basic facts about the classical Maslov index are recalled. The domain $\mathcal{D}$ is the *Siegel disc* and its Shilov boundary is realized as the *Lagrangian manifold*, and connection is made, for this example, with the presentation of the generalized Maslov index in [4] (see also [13] for earlier work in this direction). Kashiwara's definition of the Maslov index, valid for any triple of Lagrangians is recalled. This section is meant both as a model and as a preparation for the rest of the paper.

In Section 3, the definition of the generalized Maslov index for mutually transverse triples in the Shilov boundary of a (general) tube-type domain $\mathcal{D}$ is presented following [4]. The definition of the Maslov index is extended by using an arbitrary representation of the associated Euclidean Jordan algebra and shown to be independent of the representation used.

The three remaining classical series (the first one has been treated in Section 2) are treated in the next few sections. A specific realization of the domain $\mathcal{D}$ is used and the corresponding realization of the Shilov boundary S is described. For the algebra $\mathrm{Herm}(r, \mathbb{C})$ (Section 4) and for the algebra $\mathrm{Herm}(r, \mathbb{H})$ (Section 5), the extension of the definition of the Maslov index is obtained by using a standard representation of the Jordan algebra. This allows us to give a definition of the Maslov index *à la Kashiwara*, in fact very close to the real case.

Sections 6–8 deal with the Lorentz algebra $\mathcal{L}_p$. This is the most delicate part of the paper. Classical realizations of the associated domain are presented in Section 6 (one is the *Lie ball* realization), together with a description of the Shilov boundary S. The open orbits of the action of $G = \mathrm{SO}_0(p, 2)$ on $S \times S \times S$ are characterized. Although elementary, these geometrical results seem to be new. The representations of $\mathcal{L}_p$ are related to theory of Clifford modules (see [3]), and hence to spinor spaces. In Section 7, we introduce a new realization of the Shilov boundary S as a special orbit of G (or rather the spin group) in the Lagrangian manifold of the spinor space corresponding to the signature $(p, 2)$. In Section 8, a concrete expression of the extended Maslov index is obtained also in this case.

2. The Siegel disc, the Lagrangian manifold and Kashiwara's Maslov index

Let (E, A) be a real symplectic vector space of dimension $2r$. Let

$$\mathfrak{g} = \mathfrak{sp}(E) = \{X \in \mathrm{End}(E),\, A(Xv, v') + A(v, Xv') = 0\}$$

be the Lie algebra of the symplectic group $G = \mathrm{Sp}(E)$.

Choose $J \in \mathfrak{g}$ such that $J^2 = -\mathrm{Id}$ and such that the bilinear form $B(v, v') = A(v, Jv')$ (which is automatically symmetric) is positive-definite. This can be done by using a symplectic basis of E (see [14] for details). Associated to the choice of J is a Cartan decomposition of $\mathfrak{g}$. In fact denote by $\mathrm{Sym}(B)$ (resp. $\mathrm{Skew}(B)$) the space of symmetric (resp. skew-symmetric) endomorphisms of E with respect to B. Then let

$$\mathfrak{k} = \mathfrak{g} \cap \mathrm{Skew}(B) = \{X \in \mathfrak{g} \mid JX = XJ\}$$

and

$$\mathfrak{p} = \mathfrak{g} \cap \mathrm{Symm}(B) = \{X \in \mathfrak{g} \mid JX = -XJ\}.$$

Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition of $\mathfrak{g}$. Moreover, the element $H_0 = (1/2)J$ is in the center of $\mathfrak{k}$, and for $X \in \mathfrak{p}$

$$(\mathrm{ad}\, H_0)^2 = \tfrac{1}{2}\mathrm{ad}\, H_0(JX - XJ) = \tfrac{1}{4}(J(JX - XJ) - (JX - XJ)J) = -X$$

so that $\mathrm{ad}\, H_0$ defines a complex structure on $\mathfrak{p}$. The (non-compact) Riemannian symmetric pair $(\mathfrak{g}, \mathfrak{k})$ is of Hermitian type.

Let $\mathbb{E}$ be the complexification of E, extend J as a complex linear map of $\mathbb{E}$, and consider

$$\mathbb{V}^0_+ = \{v \in \mathbb{E} \mid Jv = \mathrm{i}v\}, \qquad \mathbb{V}^0_- = \{v \in \mathbb{E} \mid Jv = -\mathrm{i}v\}.$$

The spaces $\mathbb{V}^0_+$ and V^0_- are totally isotropic for (the complexification of) A (or B) and the restriction of A (or B) to $\mathbb{V}^0_+ \times V^0_-$ induces a non-degenerate duality.

Let $\mathfrak{g}_{\mathbb{C}}$ be the complexified Lie algebra of $\mathfrak{g}$ (viewed as a complex subalgebra of $\mathrm{End}(\mathbb{E})$), and define similarly $\mathfrak{k}_{\mathbb{C}}$ and $\mathfrak{p}_{\mathbb{C}}$. Let

$$\mathfrak{p}_{\pm} = \{X \in \mathfrak{p}_{\mathbb{C}} \mid \mathrm{ad}(H_0)X = \pm \mathrm{i}X\}.$$

Let $X \in \mathfrak{p}_+$. Then $JX - XJ = 2\mathrm{i}X$. So if $v \in \mathbb{V}^0_+$, then

$$JXv = (XJ + 2\mathrm{i}X)v = 3\mathrm{i}Xv$$

so that $Xv = 0$. If $v \in \mathbb{V}^0_-$, then

$$JXv = (XJ + 2\mathrm{i}X)v = \mathrm{i}Xv$$

so that X maps $\mathbb{V}^0_-$ to $\mathbb{V}^0_+$. Moreover, for $v, w \in \mathbb{V}^0_-$:

$$B(Xv, w) = A(Xv, Jw) = -A(v, XJw) = A(v, JXw) = B(v, Xw)$$

so that the induced map from $\mathbb{V}^0_-$ to $\mathbb{V}^0_+$ is symmetric with respect to the duality. The converse statement is easily verified, so that $\mathfrak{p}_+$ can be identified with the space (denoted by $\mathrm{Sym}(\mathbb{V}^0_-, \mathbb{V}^0_+)$) of homomorphisms of $\mathbb{V}^0_-$ into $\mathbb{V}^0_+$ which are symmetric with respect to the duality induced by B.

The formula

$$h(v, v') = \mathrm{i}A(\bar{v}, v')$$

defines a Hermitian form[2] on $\mathbb{E}$. It is easily verified that $\mathbb{E} = \mathbb{V}_+^0 \oplus \mathbb{V}_-^0$ is an orthogonal decomposition w.r.t. h, and that $h \mid \mathbb{V}_+^0 \times \mathbb{V}_+^0$ (resp. $h \mid \mathbb{V}_-^0 \times \mathbb{V}_-^0$) is positive-definite (resp. negative-definite). It shows in particular that h has signature (r, r).

Let us consider the set $\mathcal{D}$ of all r-dimensional complex linear spaces $\mathbb{V}_-$ of $\mathbb{E}$ such that

$$A_{|\mathbb{V}_-\times\mathbb{V}_-} = 0, \qquad h_{|\mathbb{V}_-\times\mathbb{V}_-} \ll 0. \tag{1}$$

The set $\mathcal{D}$ is a submanifold of the Grassmannian of r-dimensional subspaces in $\mathbb{E}$, and hence a complex manifold. After complexifying its action, G acts on $\mathcal{D}$. This action turns out to be transitive on $\mathcal{D}$, and $\mathcal{D}$ is the Hermitian symmetric space associated to the simple Lie group G.

The space $\mathcal{D}$ can be realized as a bounded symmetric domain as follows. If $\mathbb{V}_-$ is any subspace satisfying (1), then the restriction to $\mathbb{V}_-$ of the orthogonal projection on $\mathbb{V}_-^0$ is an isomorphism (as $\mathbb{V}_- \cap \mathbb{V}_+^0 = \{0\}$), and hence there exists a linear operator z from $\mathbb{V}_-^0$ to $\mathbb{V}_+^0$ such that

$$\mathbb{V}_- = \mathbb{V}_-^z = \{v_- + zv_- \mid v_- \in \mathbb{V}_-^0\}. \tag{2}$$

Moreover, the map z is symmetric for the duality on $\mathbb{V}_-^0 \times \mathbb{V}_+^0$ induced by A, in other words z belongs to $\mathrm{Sym}(\mathbb{V}_-^0, \mathbb{V}_+^0)$. Finally, for $z \in \mathrm{Sym}(\mathbb{V}_-^0, \mathbb{V}_+^0)$, the space $\mathbb{V}_-^z$ defined by (2) satisfies the conditions (1) if and only if

$$\forall\, v_- \in \mathbb{V}_-^0 \quad h(zv_-, zv_-) < -h(v_-, v_-)$$

so if and only if z is a contraction from $(\mathbb{V}_-^0, -h)$ to $(\mathbb{V}_+^0, h)$.

For a more explicit realization, choose an orthonormal basis $(e_1, e_2, \dots, e_r)$ of $\mathbb{V}_+$ and for $1 \le j \le r$, let $e_{j+r} = \overline{e_j}$. Then $(e_{r+1}, \dots, e_{2r})$ is an orthogonal basis of $\mathbb{V}_-^0$.

Still denote by z the matrix of this operator with respect to these basis of $\mathbb{V}_-^0$ and $\mathbb{V}_+^0$. The conditions on z which correspond to the conditions (1) on $\mathbb{V}_-$ turn out to be

$$z \in \mathrm{Sym}(r, \mathbb{C}), \qquad 1 - z\bar{z} \gg 0. \tag{3}$$

Thus $\mathcal{D}$ is realized as a the unit ball (for the spectral norm) in $\mathrm{Sym}(\mathbb{C})$. In this realization $\mathcal{D}$ is called the *Siegel disc*. The Shilov boundary of $\mathcal{D}$ is

$$S = \{\sigma \in \mathrm{Sym}(r, \mathbb{C}) \mid \bar{\sigma} = \sigma^{-1}\}$$

(see [8]).

Viewing σ as an element in $\mathrm{Sym}(\mathbb{V}_-^0, \mathbb{V}_+^0)$, this means that σ is a unitary map from $(\mathbb{V}_-^0, -h)$ to $(\mathbb{V}_+^0, h)$. Going back to the geometric realization, the Shilov boundary of $\mathcal{D}$ consists in the r-dimensional complex vector spaces of the form

$$\mathbb{W}^\sigma = \{v_- + \sigma v_- | v_- \in \mathbb{V}_-^0\} \tag{4}$$

for $\sigma \in S$. Such a space $\mathbb{W}$ satisfies the conditions:

$$A_{|\mathbb{W}\times\mathbb{W}} = 0, \qquad h_{|\mathbb{W}\times\mathbb{W}} = 0 \tag{5}$$

[2] Hermitian forms are assumed to be $\mathbb{C}$-linear in the second variable.

and conversely, any r-dimensional subspace $\mathbb{W}$ of $\mathbb{E}$ which satisfies (5) can be written as $\mathbb{W}^\sigma$ for some $\sigma \in S$. The conditions (5) imply that $\bar{\mathbb{W}}$ is A-orthogonal to $\mathbb{W}$, and hence $\bar{\mathbb{W}} = \mathbb{W}$. So $\mathbb{W}$ is the complexification of the (real) Lagrangian subspace $W = \mathbb{W} \cap E$ of E. Conversely, if W is a Lagrangian subspace of E, then its complexification $\mathbb{W}$ satisfies (5). This shows that the Shilov boundary S of $\mathcal{D}$ can be identified with the (real) *Lagrangian manifold* of all Lagrangian spaces in E.

A more explicit expression of the identification $\sigma \mapsto W^\sigma = \mathbb{W}^\sigma \cap E$ will be needed, at least when σ is diagonal. For $1 \le j \le r$, set

$$f_j = \frac{e_j + e_{j+r}}{\sqrt{2}}, \qquad f_{j+r} = \frac{-e_j + e_{j+r}}{\mathrm{i}\sqrt{2}}.$$

Then $\{f_j\}_{1\le j\le 2r}$ is a symplectic basis of W. For $1 \le j \le r$, let $\lambda_j = \mathrm{e}^{\mathrm{i}\theta_j}$ be a complex number of modulus 1, and consider

$$\sigma = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_r \end{pmatrix}.$$

The associated Lagrangian W^σ is the subspace:

$$W^\sigma = \bigoplus_{1\le j\le r} \mathbb{R}(\lambda_j^{1/2} e_j + \lambda_j^{-1/2} e_{j+r}) = \bigoplus_{1\le j\le r} \mathbb{R}\left(\cos\left(\frac{\theta_j}{2}\right) f_j + \sin\left(\frac{\theta_j}{2}\right) f_{j+r}\right). \tag{6}$$

Transversality of two Lagrangian spaces W_1, W_2 is denoted by

$$W_1 \top W_2 \Leftrightarrow W_1 \cap W_2 = \{0\}.$$

For $\sigma, \tau \in S$, let $W^\sigma = \mathbb{W}^\sigma \cap E$ and $W^\tau = \mathbb{W}^\tau \cap E$. Then the relation $W^\sigma \top W^\tau$ is equivalent to $\sigma - \tau$ being injective or equivalently to $\mathrm{Det}(\sigma - \tau) \ne 0$. This condition is denoted by $\sigma \top \tau$. The symplectic group G acts on pairs of transverse Lagrangian spaces, and this action is transitive.

Now consider three mutually transverse Lagrangian spaces (W_1, W_2, W_3) in E. There exists a *normal form* for the triple. More precisely, there exists a symplectic basis $\{f_1, f_2, \dots, f_r, f_{r+1}, \dots, f_{2r}\}$ of E such that

$$\begin{aligned} W_1 &= \mathbb{R} f_1 \oplus \mathbb{R} f_2 \oplus \cdots \oplus \mathbb{R} f_r, \qquad W_2 = \mathbb{R} f_{r+1} \oplus \mathbb{R} f_{r+2} \oplus \cdots \oplus \mathbb{R} f_{2r}, \\ W_3 &= \mathbb{R}(f_1 + f_{r+1}) \oplus \cdots \oplus \mathbb{R}(f_k + f_{r+k}) \\ &\quad \oplus \mathbb{R}(f_{k+1} - f_{r+k+1}) \oplus \cdots \oplus \mathbb{R}(f_r - f_{2r}), \end{aligned} \tag{7}$$

where k is an integer, $0 \le k \le r$ (see [11, Corollary 1.5.7]). Denote by $S^3_\top$ the set of triples of mutually transverse Lagrangian spaces in E. The existence of a normal form implies that there are exactly $r + 1$ orbits in $S^3_\top$ under the action of G.

Kashiwara (see [11, Section 1.5]) proposed a definition of the Maslov index for any triple of Lagrangian spaces (not necessarily mutually transverse). Let W_1, W_2, W_3 be three Lagrangian spaces in E. On $W_1 \times W_2 \times W_3$, consider the quadratic form Q defined by

$$Q(v_1, v_2, v_3) = A(v_1, v_2) + A(v_2, v_3) + A(v_3, v_1)$$

and defines the Maslov index by

$$\iota(W_1, W_2, W_3) = \operatorname{sgn}(Q),$$

where sgn stands for the signature of the quadratic form. It is by construction invariant by G, hence constant on the orbits. For a triple in $S^3_\top$, the Maslov index is shown to be equal to $r - 2k$, where k is the integer which appears in the normal form of the three mutually transverse Lagrangian spaces (see [11, Corollary 1.5.7]).

A slightly more general formula for the Maslov index (covering some non-mutually transverse cases) will be needed.

Lemma 2.1. *Let $\{f_1, f_2, \dots, f_r, f_{r+1}, \dots, f_{2r}\}$ be a symplectic basis of E. For $1 \le j \le r$, let $\theta_j \in \mathbb{R}/2\pi\mathbb{Z}$ and consider the three following Lagrangian spaces*

$$W_1 = \mathbb{R}f_1 \oplus \mathbb{R}f_2 \oplus \cdots \oplus \mathbb{R}f_r, \qquad W_2 = \mathbb{R}f_{r+1} \oplus \mathbb{R}f_{r+2} \oplus \cdots \oplus \mathbb{R}f_{2r},$$
$$W_3 = \bigoplus_{1 \le j \le r} \mathbb{R}\left(\cos\left(\frac{\theta_j}{2}\right) f_j + \sin\left(\frac{\theta_j}{2}\right) f_{j+r}\right).$$

Then

$$\iota(W_1, W_2, W_3) = \sharp\{j \mid \sin\theta_j < 0\} - \sharp\{j \mid \sin\theta_j > 0\}. \tag{8}$$

As W_1 and W_2 are transverse, the proof of (8) is a consequence of [11, Lemma 1.5.4].

These results can be transferred to S. For $(\sigma_1, \sigma_2, \sigma_3) \in S^3$, set

$$\iota(\sigma_1, \sigma_2, \sigma_3) = \iota(W^{\sigma_1}, W^{\sigma_2}, W^{\sigma_3}).$$

Lemma 2.2. *For $1 \le j \le r$, let $\theta_j \in \mathbb{R}/2\pi\mathbb{Z}$. Let*

$$\sigma = \begin{pmatrix} e^{i\theta_1} & 0 & \cdots & 0 \\ 0 & e^{i\theta_2} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & e^{i\theta_r} \end{pmatrix}.$$

Then

$$\iota(\mathrm{Id}_r, -\mathrm{Id}_r, \sigma) = \sharp\{j \mid \sin\theta_j < 0\} - \sharp\{j \mid \sin\theta_j > 0\}.$$

The Maslov index is skew-symmetric with respect to permutations of the three arguments, namely:

$$\iota(\sigma_{\pi(1)}, \sigma_{\pi(2)}, \sigma_{\pi(3)}) = \varepsilon(\pi)\iota(\sigma_1, \sigma_2, \sigma_3) \tag{9}$$

for any permutation π of the set $\{1, 2, 3\}$ of signature $\varepsilon(\pi) = \pm 1$. The Maslov index satisfies the cocycle relation:

$$\iota(\sigma_1, \sigma_2, \sigma_3) = \iota(\sigma_1, \sigma_2, \sigma_4) + \iota(\sigma_2, \sigma_3, \sigma_4) + \iota(\sigma_3, \sigma_1, \sigma_4) \tag{10}$$

for any four elements $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ of S.

3. The Maslov index on the Shilov boundary of a classical bounded domain of tube-type

The present author in a joint work with Ørsted (see [4,5]) proposed a generalization of the Maslov index on the Shilov boundary of any bounded domain of tube-type.

Recall there are two main equivalent approaches to Hermitian symmetric domains. One uses Lie theory, the other uses the theory of Jordan triples. Both of them will be considered. Main references are [7,8,10,12,14].

Let $\mathfrak{g}$ be a simple real Lie algebra of the non-compact type. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition. Then the corresponding symmetric space is of Hermitian type if and only if there exists an element H_0 (then unique up to a sign) in the center of $\mathfrak{k}$, such that $(\mathrm{ad}\, H_{0|\mathfrak{p}})^2 = -\mathrm{Id}_{\mathfrak{p}}$. Hence ad H_0 induces a complex structure on $\mathfrak{p}$. Let $\mathfrak{g}_{\mathbb{C}}$ be the complexified Lie algebra of $\mathfrak{g}$ and denote by σ the conjugation of $\mathfrak{g}_{\mathbb{C}}$ with respect to the compact real form $\mathfrak{g} = \mathfrak{k} \oplus \mathrm{i}\mathfrak{p}$. Let $\mathfrak{p}_{\mathbb{C}}$ be the complexification of $\mathfrak{p}$, and let

$$\mathfrak{p}_{\pm} = \{X \in \mathfrak{p}_{\mathbb{C}} | \mathrm{ad}\, H_0(X) = \pm \mathrm{i}X\}.$$

The space $\mathfrak{p}_+$ is a commutative subalgebra of $\mathfrak{g}_{\mathbb{C}}$ and has a structure of *positive-definite Hermitian Jordan triple system*. The triple product is given by

$$\{x, y, z\} = \tfrac{1}{2}[[x, \sigma y], z].$$

Observe that it is $\mathbb{C}$-linear in x and z, and $\mathbb{C}$-conjugate linear in y.

For $x, y \in \mathfrak{p}_+$, let $x\Box y$ be the endomorphism of $\mathfrak{p}_+$ given by

$$(x\Box y)z = \{x, y, z\}.$$

Then the Hermitian symmetric space is realized à la Harish Chandra as

$$\mathcal{D} = \{z \in \mathfrak{p}_+ |\, \|z\Box z\|_{\mathrm{op}} < 1\},$$

where $\mathfrak{p}_+$ is equipped with a certain inner product constructed from the Killing form of $\mathfrak{g}$, and $\| \, \|_{\mathrm{op}}$ is the operator norm with respect to this inner product.

The Shilov boundary S of the domain $\mathcal{D}$ is

$$S = \{\sigma \in \mathfrak{p}_+ | \{\sigma, \sigma, \sigma\} = \sigma\}.$$

The domains of tube-type correspond to the case where $\mathfrak{p}_+$ has a structure of *positive-definite Hermitian Jordan algebra*. This occurs if and only if there exists an element e in $\mathfrak{p}_+$ such that

$$\{e, e, z\} = z \quad \forall z \in \mathfrak{p}_+$$

or in Lie terms

$$[e, \sigma(e)] = 2\mathrm{i}H_0.$$

The real subspace $J = \{z \in \mathfrak{p}_+ | \{e, z, e\} = z\}$ is then a Euclidean Jordan algebra for the Jordan multiplication

$$x \cdot y = \{x, e, y\}$$

with e as unit element.

The domain $\mathcal{D}$ is realized à la *Harish Chandra* as the unit ball (in the complexification $\mathbb{J}$ of J) for a certain norm, called the spectral norm.

The process can be reversed, by using the *Koecher–Kantor–Tits construction* to recover the Lie algebra $\mathfrak{g}$ (see [14]). For tube-type domains, a more global approach is presented in [8], which notation and results are freely used in the sequel.

Let J be a simple Euclidean Jordan algebra. Let e be the unit in J, Ω the (open) cone of squares in J, and det the determinant polynomial (also called norm). Let $\mathbb{J}$ be the complexification of J and denote by $|\ |_{\mathrm{op}}$ the *spectral norm* on $\mathbb{J}$. The corresponding unit ball

$$\mathcal{D} = \mathcal{D}_E = \{z \in \mathbb{J} |\ |z|_{\mathrm{op}} < 1\}$$

is a bounded symmetric domain of tube-type. It is holomorphically equivalent to the tube domain $\mathcal{T} = J + \mathrm{i}\Omega \subset \mathbb{J}$ (this is why $\mathcal{D}$ is said to be of tube-type). Let $G = \mathrm{Hol}(\mathcal{D})^0$ be the neutral component of the group of holomorphic diffeomorphisms of $\mathcal{D}$.

The Shilov boundary S of $\mathcal{D}$ is described as

$$S = \{\sigma \in \mathbb{J} | \bar{z} = z^{-1}\}.$$

Two elements σ, τ of S are said to be *transverse* (see [4]) if and only if

$$\sigma \top \tau \overset{\mathrm{def}}{\Leftrightarrow} \det(\sigma - \tau) \neq 0.$$

The group G preserves the tranversality, and acts transitively on

$$S^2_\top = \{(\sigma, \tau) \in S \times S | \sigma \top \tau\}.$$

Let

$$S^3_\top = \{(\sigma_1, \sigma_2, \sigma_3) \in S \times S \times S | \sigma_j \top \sigma_k \text{ for } j \neq k\}.$$

There are a finite number of orbits under G in $S^3_\top$ (exactly $r + 1$ where r is the rank of the Jordan algebra J and also the rank of $\mathcal{D}$ as a Riemannian symmetric space). Representatives of the orbits can be described. To each triple of mutually transverse elements of S is associated its *Maslov index* $\iota(\sigma_1, \sigma_2, \sigma_3)$. The Maslov index is invariant by G, satisfies the skew-symmetry property (9) and the cocycle relation (10). The Maslov index characterizes the orbits of G in $S^3_\top$ in the sense that two triples are conjugate under G if and only if they have the same Maslov index.

The question now to be addressed is the existence of an extension of the Maslov index to all triples, including triples which do not satisfy the tranversality condition. When $J =$

$\mathrm{Sym}(r, \mathbb{R})$, the situation was studied in the first section, and so there is indeed an extension, given by Kashiwara's formula.

For a classical Euclidean Jordan algebra the question can be answered by using a representation of the Jordan algebra.

Let $(F, (\cdot|\cdot))$ be a Euclidean vector space, of dimension N. A representation of J in F is a Jordan algebra homomorphism $\Phi : J \to \mathrm{Sym}(F)$. For any $x \in J$, $\Phi(x)$ is an endomorphism of F such that for all $x, y \in J$, $\lambda \in \mathbb{R}$, $\xi, \eta \in F$:

(i) $\Phi(x + \lambda y) = \Phi(x) + \lambda\Phi(y)$.
(ii) $\Phi(xy) = \frac{1}{2}(\Phi(x)\Phi(y) + \Phi(y)\Phi(x))$, $\Phi(e) = \mathrm{Id}_E$.
(iii) $(\Phi(x)\xi|\eta) = (\xi|\Phi(x)\eta)$.

Two consequences of the axioms are

$$\Phi(P(x)y) = \Phi(x)\phi(y)\Phi(x) \quad \text{for all} \quad x, y \in J \tag{11}$$

(P is the *quadratic representation* of J), and

$$\Phi(x) \gg 0 \quad \text{for} \quad x \in \Omega. \tag{12}$$

If $e = \sum_{j=1}^{r} c_j$ is a Peirce decomposition of the unit of J, then

$$\mathrm{Id}_F = \sum_{j=1}^{r} \Phi(c_j)$$

is an orthogonal decomposition of the identity in F. For any $j, 1 \le j \le r$, $\Phi(c_j)$ is an orthogonal projection, and its rank is easily seen to be independent of j (see, e.g. [3]), so that $d =: (N/r)$ is an integer.

Let $\mathbb{F}$ be the complexification of F and still denote by Φ the complex extension of Φ. As Φ maps idempotents of J to idempotents of $\mathrm{Sym}(F)$, one verifies that for any $z \in \mathbb{J}$:

$$|\Phi(z)|_{\mathrm{op}} = |z|_{\mathrm{op}}$$

so that Φ maps $\mathcal{D}$ into the Siegel disc $\mathcal{D}_F = \{z \in \mathrm{Sym}(\mathbb{F}) | \mathrm{Id}_{\mathbb{F}} - z\bar{z} \gg 0\}$.

Moreover, Φ sends the Shilov boundary S of $\mathcal{D}$ into the Shilov boundary S_F of the Siegel disc, preserves transversality and satisfies the following relation (proved in even greater generality in [4, Theorem 6.3]):

$$\forall(\sigma_1, \sigma_2, \sigma_3) \in S_{\top}^3 \quad \iota(\Phi(\sigma_1), \Phi(\sigma_2), \Phi(\sigma_3)) = \frac{N}{r}\iota(\sigma_1, \sigma_2, \sigma_3). \tag{13}$$

Now the left-hand side of the formula is defined (by Kashiwara's formula) for any triple. Take as a definition of the extended Maslov index the formula:

$$\iota(\sigma_1, \sigma_2, \sigma_3) = \frac{r}{N}\iota(\Phi(\sigma_1), \Phi(\sigma_2), \Phi(\sigma_3)). \tag{14}$$

Theorem 3.1. *The extended Maslov index as defined by* (14) *is independent of the representation Φ used for its definition. It takes integral values between $-r$ and r, is invariant by G, skew-symmetric with respect to the permutations of the three indices, and satisfies the cocyle relation* (10).

The proof is divided into several steps.

Step 1. Let us first prove the invariance by G. From the representation Φ, it is possible to construct a Lie algebra homomorphism of $\mathfrak{g} = \mathrm{Lie}(G)$ into the Lie algebra $\mathfrak{sp}(E \times E)$, to which it corresponds a (local) homomorphism of G into the symplectic group, which could be used to prove the invariance. There is a more global approach, for which no reference seems to be available. Hence a couple of lemmas are needed.

Denote by $\mathrm{Str}(J)$ the *structure group* of J. Define

$$H = \{(l, g) \in \mathrm{GL}(J) \times \mathrm{GL}(F) \mid \Phi(lx) = g\Phi(x)g'\}.$$

Clearly H is a closed subgroup of $\mathrm{GL}(J) \times \mathrm{GL}(F)$.

Lemma 3.2. *Let* $(l, g) \in H$. *Then* $l \in \mathrm{Str}(J)$.

Proof. Let x be in J. For all $y \in J$:

$$\Phi(P(lx)y) = \phi(lx)\Phi(y)\Phi(lx) = g\Phi(x)g'\Phi(y)g\Phi(x)g'.$$

Assume further that x is invertible. This last identity can be rewritten as

$$\Phi(P(x^{-1})l^{-1}P(lx)y) = g'\Phi(y)g.$$

But as J is assumed to be simple, Φ is injective, and hence $P(x^{-1})l^{-1}P(lx)$ is independent of x, so there exists a certain element $h \in \mathrm{GL}(J)$ such that

$$P(lx) = lP(x)h$$

for any invertible element x, and hence by continuity for any $x \in J$. From [8, Lemma VIII.2.3], this implies that $h = l'$ and $l \in \mathrm{Str}(J)$. □

Lemma 3.3. *Let* $l \in \mathrm{Str}(J)^0$. *Then there exists* $g \in \mathrm{GL}(F)$ *such that* (h, g) *belongs to* H.

Let $x \in J$ be invertible. Then $P(x) \in \mathrm{Str}(J)$. Formula (11) shows that $(P(x), \Phi(x))$ belongs to H. But the subgroup of $\mathrm{Str}(J)$ generated by the $\{P(x), x \in J\}$ (called the *inner structure group* in [15]) contains the neutral component of the group $\mathrm{Str}(J)$. Hence the projection on the first factor, from H into $\mathrm{Str}(J)$, contains $\mathrm{Str}(J)^0$ in its image.

Recall that $\mathcal{T}_\Omega = J + \mathrm{i}\Omega$ is holomorphically equivalent to the domain $\mathcal{D}$. If $y \in \Omega$, then $\Phi(y)$ is positive-definite, so that Φ maps $\mathcal{T}_\Omega$ into the Siegel half-space

$$\mathcal{T}_F = \{s + \mathrm{i}t | s, t \in \mathrm{Sym}(F), t \gg 0\}.$$

The group $\mathrm{Hol}(T)^0$ (=connected component of the identity in the group of all holomorphic diffeomorphisms of $\mathcal{T}$) is isomorphic to G, but is easier to handle (a family of generators is easy to describe). In fact, let us consider the group generated by

- the translations $t_v : z \mapsto z + v$, for $v \in J$;
- the transformations $z \mapsto lz$, where $l \in \mathrm{Str}(J)^0$;
- the inversion $\iota : z \mapsto -z^{-1}$.

All these transformations belong to $\mathrm{Hol}(\mathcal{T}_\Omega)$ and the group they generated contains $\mathrm{Hol}(\mathcal{T}_\Omega)^0$ (see [8, Theorem X.5.6]).

Let

$$C = \{(h, g) \in \mathrm{Hol}(\mathcal{T}_\Omega) \times \mathrm{Hol}(\mathcal{T}_F) | \Phi(h(z)) = g(\Phi(z))\}.$$

Clearly C is a closed subgroup of $\mathrm{Hol}(\mathcal{T}_\Omega) \times \mathrm{Hol}(\mathcal{T}_F)$.

Lemma 3.4. *Let $h \in \mathrm{Hol}(\mathcal{T}_\Omega)^0$. Then there exists $g \in \mathrm{Hol}(\mathcal{T}_F)$ such that $(h, g) \in C$.*

Proof. For $h = t_v$ for some $v \in J$, one has $\Phi(z+v) = \Phi(z)+\Phi(v)$, and hence $(t_v, t_{\Phi(v)}) \in C$. For $l \in \mathrm{Str}(J)^0$, there exists by Lemma 3.2, $g \in \mathrm{GL}(F)$ such that $(l, g) \in H$. By complexification:

$$\Phi(lz) = g\Phi(z)g' \quad \forall z \in \mathbb{J}.$$

Hence $(l, g) \in C$. Finally, as Φ is a homomorphism of Jordan algebras, $\Phi(-z^{-1}) = -\Phi(z)^{-1}$, and hence $(\iota, \iota_{\mathrm{Sym}(F)}) \in C$. As the group generated by all these transformations contains $\mathrm{Hol}(\mathcal{T}_\Omega)^0$, the lemma follows.

Use a Cayley transform to conclude that for any element $l \in G = \mathrm{Hol}(\mathcal{D})^0$, there exists an element $g \in \mathrm{Hol}(\mathcal{D}_F)$ such that $\Phi(l(z)) = g(\Phi(z))$ for all $z \in \mathcal{D}$. Recalling that the group $\mathrm{Hol}(\mathcal{D}_F)$ is isomorphic to the symplectic group of $F \times F$ and that the map $\sigma \mapsto W^\sigma$ is equivariant with respect to this isomorphism, this last result implies that the extended Maslov index is invariant by G. □

Step 2. The independence from the representation Φ will be a consequence of a computation of the extended Maslov index for triples in which only one couple is not assumed to be transverse.

Fix a Peirce decomposition of the unit $e = \sum_{j=1}^{r} c_j$. For p, q such that $p \geq 0$, $q \geq 0$, $p + q \leq r$, let

$$\varepsilon_{p,q} = -\mathrm{i}\left(\sum_{j=1}^{p} c_j\right) + \mathrm{i}\left(\sum_{j=p+1}^{p+q} c_j\right) - \sum_{j=p+q+1}^{r} c_j.$$

Observe that $\varepsilon_{p,q}$ belongs to S and $\varepsilon_{p,q} \top e$.

Lemma 3.5. *Let $(\sigma_1, \sigma_2, \sigma_3) \in S^3$, and assume that $\sigma_1 \top \sigma_2$ and $\sigma_1 \top \sigma_3$. Then there exists two integers p, q with $p \geq 0$, $q \geq 0$, $p + q \leq r$ and an element $g \in G$ such that*

$$(\sigma_1, \sigma_2, \sigma_3) = (g(e), g(-e), g(\varepsilon_{p,q})).$$

The proof is completely similar to the proof of Theorem 4.3 in [4]. As G acts transitively on $S^2_\top$, it is possible to assume that $\sigma_1 = e$, $\sigma_2 = -e$. As σ_3 is also transverse to e, the Cayley transform $c(z) = \mathrm{i}(e + z)(e - z)^{-1}$ is well defined at σ_3. Set $\xi_3 = c(\sigma_3) \in J$. The image of the stabilizer of $(e, -e)$ under the Cayley transform is $L = \mathrm{Str}(J)^0$. Now any element of J is conjugate under L to

$$\sum_{j=1}^{p} c_j - \sum_{j=p+1}^{p+q} c_j$$

for some integers p, q with $p \geq 0$, $q \geq 0$, $p + q \leq r$. Apply the inverse Cayley transform to get the lemma.

The image through Φ of the triple $(\sigma_1, \sigma_2, \sigma_3)$ is conjugate under the symplectic group to the triple $(\mathrm{Id}_F, -\mathrm{Id}_F, E_{p,q})$ where

$$E_{p,q} = -\mathrm{i}\left(\sum_{j=1}^{p} \Phi(c_j)\right) + \mathrm{i}\left(\sum_{j=p+1}^{p+q} \Phi(c_j)\right) - \sum_{j=p+q+1}^{r} \Phi(c_j).$$

All three maps $(\mathrm{Id}_F, -\mathrm{Id}_F, E_{p,q})$ are diagonal in the basis $\{f_j, 1 \leq j \leq N\}$. Lemma 3.2 can be used to compute the corresponding Maslov index, namely:

$$\iota(W^{\mathrm{Id}}, W^{-\mathrm{Id}}, W^{E_{p,q}}) = (p - q)d.$$

Hence the definition of the extended Maslov index gives

$$\iota(\sigma_1, \sigma_2, \sigma_3) = \iota(e, -e, \varepsilon_{p,q}) = (p - q).$$

So for these triples, the value of the extended Maslov index is an integer and does not depend on the representation used for the definition.

Now let $(\sigma_1, \sigma_2, \sigma_3) \in S^3$ be an arbitrary triple. Then choose an element $\sigma_4 \in S$ such that σ_4 is transverse to all $\sigma_1, \sigma_2, \sigma_3$. Then, thanks to the cocycle relation (10) for the classical Maslov index, one has

$$\iota(\sigma_1, \sigma_2, \sigma_3) = \iota(\sigma_1, \sigma_2, \sigma_4) + \iota(\sigma_2, \sigma_3, \sigma_4) + \iota(\sigma_3, \sigma_1, \sigma_4).$$

As σ_4 is transverse to the two other elements of any triple in the right-hand side of the formula, these three terms are of the type already considered. Hence, the left-hand side is an integer and does not depend on the particular representation considered.

Step 3. The extended Maslov is skew-symmetric with respect to permutations of the indices $\{1, 2, 3\}$, and satisfies the cocycle relation, as these properties are true for the classical Maslov index. This completes the proof of Theorem 4.1.

4. The unitary case

Let V a vector space over $\mathbb{C}$ of dimension $2r$, and let h be a skew-Hermitian[3] form h of signature (r, r). Denote by $G = U(V, h) \simeq U(r, r)$ the subgroup of linear transformations

[3] The use of a skew-Hermitian form rather than a Hermitian form is to stress the analogy with the real and quaternionic case.

preserving the form h. Let V_+^0 and V_-^0 be subspaces such that

$$V = V_+^0 \oplus V_-^0, \qquad V_+^0 \perp V_-^0, \qquad \mathrm{i}h|V_+^0 \gg 0, \qquad \mathrm{i}h|V_-^0 \ll 0.$$

Denote by p_+ (resp. p_-) the orthogonal projection on V_+^0 (resp. V_-^0).

Let us consider the set $\mathcal{D}$ of all r-dimensional subspaces V_- of V such that $\mathrm{i}h_{|V_-} \ll 0$. It is an open set in the Grassmannian of r-dimensional subspaces of V, on which G acts and $\mathcal{D}$ is a realization of its associated Hermitian symmetric space (see [14, Appendix]).

Let $V_- \in \mathcal{D}$. As $V_+^0 \cap V_- = \{0\}$, the restriction of p_- to V_- is a linear bijection of V_- on V_-^0, and hence there exists a linear map $z : V_-^0 \to V_+^0$ such that

$$V_- = V_-^z = \{\xi + z\xi | \xi \in V_-^0\}. \tag{15}$$

Moreover, the space V_-^z defined by (15) belongs to $\mathcal{D}$ if and only if z satisfies

$$\mathrm{i}h(z\xi, z\xi) < -\mathrm{i}h(\xi, \xi)$$

for all $\xi \in V_-^0$, which means that

$$|z|_{\mathrm{op}} < 1$$

as a map from $(V_-^0, -\mathrm{i}h_{|V_-^0})$ into $(V_+^0, \mathrm{i}h_{|V_+^0})$. This shows that, by taking appropriate basis in V_-^0 and V_+^0, the Hermitian symmetric space is realized as the unit ball in $\mathrm{Mat}(r, \mathbb{C})$.

Consider the Euclidean Jordan algebra $J = \mathrm{Herm}(r, \mathbb{C})$ with the Jordan product $x \cdot y = (1/2)(xy + yx)$ and inner product $\mathrm{Re}\,\mathrm{tr}(xy)$. Its complexification can be realized as $\mathbb{J} = \mathrm{Mat}(r, \mathbb{C})$, the conjugation with respect to the real form J being $x \mapsto x^* = (\bar{x})^t$. The spectral norm coincides with the operator norm and so the associated Hermitian symmetric space is the unit ball in $\mathrm{Mat}(r, \mathbb{C})$.

In this picture, it is easy to determine the Shilov boundary of $\mathcal{D}$. It is the space of unitary matrices

$$S = \{\sigma \in \mathrm{Mat}(r, \mathbb{C}) | \bar{\sigma} = \sigma^{-1}\} \simeq U(r).$$

Transferring back to the first picture of $\mathcal{D}$, the Shilov boundary appears as the space of r-dimensional subspaces $W = W^\sigma$ in V defined by

$$W^\sigma = \{\xi + \sigma\xi | \xi \in V_-^0\},$$

where $\sigma : V_-^0 \to V_+^0$ is an isometry for the inner products $-\mathrm{i}h_{|V_-^0}$ and $\mathrm{i}h_{|V_+^0}$. The space W^σ is easily seen to be totally isotropic for h, and conversely, any maximal totally isotropic subspace of V is of the form W^σ for some unitary $\sigma : V_-^0 \to V_+^0$. Hence the Shilov boundary S of $\mathcal{D}$ can be identified with the manifold of (complex) Lagrangians in V.

The Jordan algebra $\mathrm{Herm}(r, \mathbb{C})$ has a natural representation. Let us consider $F = (\mathbb{C}^r)_{\mathbb{R}}$ the real vector space of dimension $2r$ underlying the complex vector space $\mathbb{C}^r$, and consider the Euclidean inner product defined by

$$(\xi, \eta) = \mathrm{Re}\langle \xi, \eta \rangle,$$

where $\langle\xi, \eta\rangle$ is the standard inner product on $\mathbb{C}^r$. Then, for $x \in J$ and $\xi \in F$ let

$$\Phi(x)\xi = x\xi.$$

This defines a Euclidean representation, and the index d equals $2r/r = 2$. As explained in Section 3, this representation extended to the complexification $\mathbb{J}$ can be used to extend the definition of the Maslov index.

In turn, the space $E = F \times F$ can be viewed as the real vector subspace underlying the $2r$-dimensional complex space V, the symplectic form being $\operatorname{Re} h$ (this where the choice of h as a skew-Hermitian rather than Hermitian form is justified) and the representation associates to a complex Lagrangian W in V the real Lagrangian subspace $W_{\mathbb{R}}$ in E underlying W. This leads to a definition of the extended Maslov index à la Kashiwara for triples of complex Lagrangians. For three complex Lagrangians W_1, W_2, W_3 in V, define Q on $(W_1 \times W_2 \times W_3) \times (W_1 \times W_2 \times W_3)$ by

$$Q((\xi_1, \xi_2, \xi_3), (\eta_1, \eta_2, \eta_3)) = (h(\xi_1, \eta_2) - h(\xi_2, \eta_1) + h(\xi_2, \eta_3) - h(\xi_3, \eta_2) + h(\xi_3, \eta_1) - h(\xi_1, \eta_3)). \tag{16}$$

As h is assumed to be skew-Hermitian, this defines an Hermitian form on $W_1 \times W_2 \times W_3$.

Theorem 4.1. *Let W_1, W_2, W_3 be three complex Lagrangians in V. The extended Maslov index is given by*

$$\iota(W_1, W_2, W_3) = \operatorname{sgn}(Q), \tag{17}$$

where sgn Q *is the signature of the Hermitian form Q on $W_1 \times W_2 \times W_3$ defined by* (16).

Proof. The corresponding real quadratic form (on $(W_1 \times W_2 \times W_3)_{\mathbb{R}}$) is given by

$$Q(\xi_1, \xi_2, \xi_3) = 2 \operatorname{Re} h(\xi_1, \xi_2) + 2 \operatorname{Re} h(\xi_2, \xi_3) + 2 \operatorname{Re} h(\xi_3, \xi_1).$$

The signature of this quadratic form is (Kashiwara's definition of) the classical Maslov index $\iota(W_{1\mathbb{R}}, W_{2\mathbb{R}}, W_{3\mathbb{R}})$. But the signature $\operatorname{sgn}_{\mathbb{C}}$ of Q as a Hermitian form is *half* the signature $\operatorname{sgn}_{\mathbb{R}}$ of Q viewed as a real quadratic from. Hence the extended Maslov index for W_1, W_2, W_3 as defined in Section 3 is given by

$$\iota(W_1, W_2, W_3) = \tfrac{1}{2}\iota(W_{1\mathbb{R}}, W_{1\mathbb{R}}, W_{1\mathbb{R}}) = \tfrac{1}{2} \operatorname{sgn}_{\mathbb{R}} Q = \operatorname{sgn}_{\mathbb{C}}(Q). \qquad \square \tag{18}$$

5. The quaternionic case

Let $\mathbb{H}$ be the quaternion field, with its standard involution. Quaternions will be described by two complex numbers:

$$q = a + b\mathrm{i} + cj + dk = z + wj,$$

where $z = a + \mathrm{i}b$ and $w = c + \mathrm{i}d$ for $a, b, c, d \in \mathbb{R}$. With this notation, notice that for $z, w \in \mathbb{C}$, $jz = \bar{z}j$ and $\bar{q} = \bar{z} - wj$.

Let V be a (right) vector space over $\mathbb{H}$ of dimension p, and let h be a non-degenerate *skew-Hermitian* $\mathbb{H}$-valued form on V, i.e. $h : V \times V \to \mathbb{H}$ satisfies

$$h(v, v'q) = h(v, v')q, \qquad h(vq, v') = \bar{q}h(v, v'), \qquad h(v', v) = -\overline{h(v, v')}$$

for all $q \in \mathbb{H}$ and $v, v' \in V$. The main references (and most of the notation) are from [9] (or occasionally from [14]). Now introduce the $\mathbb{C}$-valued forms A and B defined on V (viewed as a complex vector space of dimension $2p$)

$$h(v, v') = A(v, v') + jB(v, v'). \tag{19}$$

The form B is $\mathbb{C}$-bilinear and symmetric, whereas A is skew-Hermitian. Moreover, the forms A and B are related by

$$A(v, v') = -B(vj, v'). \tag{20}$$

As a consequence:

$$A(vj, v'j) = \overline{A(v, v')}$$

for $v, v' \in V$. If V_+ is a (complex) subspace of V such that $\mathrm{i}A_{|V_+\times V_+} \gg 0$, then letting $V_- = (V_+)j$ one has $\mathrm{i}A_{|V_-\times V_-} \ll 0$, so that the Hermitian form $\mathrm{i}A$ is of signature (p, p).

Let $G = U(V, h)$ be the group of $\mathbb{H}$-linear transformations which preserve the form h. It can also be looked at as the group of $\mathbb{C}$-linear transformations g which preserve the symmetric form B and such that $g(vj) = (gv)j$ for all $v \in V$. This last group is usually denoted by $O^*(2p, \mathbb{C})$.

Let us consider the p-dimensional subspaces V_- of V such that

$$B_{|V_-\times V_-} = 0, \qquad \mathrm{i}A_{|V_-\times V_-} \ll 0. \tag{21}$$

These conditions are stable by the action of G, and the space $\mathcal{D}$ of such subspaces (viewed as a subspace of the Grassmannian $G(r, 2r)$) turns out to be a Hermitian symmetric space, with G as the neutral component of its group of biholomorphic transforms. To make connections with matrix realizations of $\mathcal{D}$, fix a base point V_-^0 in $\mathcal{D}$, and observe that $V_+^0 = V_-^0 j$ is a complementary subspace, such that $B_{|V_+^0\times V_+^0} = 0$ and $\mathrm{i}A_{|V_+^0\times V_+^0} \gg 0$. If V_- is any element of $\mathcal{D}$, then the projection on V_-^0 parallel to V_+^0 is a ($\mathbb{C}$-linear) isomorphism. Hence there exists a linear map $z : V_-^0 \to V_+^0$ such that

$$V_- = V_-^z = \{\xi + z\xi, \xi \in V_-^0\}.$$

Moreover, the space V_-^z thus defined satisfies the conditions (20) are satisfied if and only if for appropriate choices of dual basis in V_-^0 and V_+^0

$$z^t = -z, \qquad I - z^*z \gg 0$$

(see [14, Appendix]). So $\mathcal{D}$ is realized as the unit ball in the space of complex $p \times p$ skew-symmetric matrices. The domain $\mathcal{D}$ is of tube-type if and only if p is even, say $p = 2r$, which will be assumed for the rest of this section.

Consider the Euclidean Jordan algebra $J = \mathrm{Herm}(r, \mathbb{H})$ with the Jordan product $x \cdot y = (1/2)(xy + yx)$. Its complexification can be described as $\mathrm{Skew}(2r, \mathbb{C})$ with a certain Jordan product whose exact formulation is not necessary for our purpose. The associated Hermitian symmetric domain is the unit ball in $\mathrm{Skew}(2r, \mathbb{C})$ (see [8] for details).

In this picture the determination of the Shilov boundary S of $\mathcal{D}$ is easy. It is given by the conditions:

$$z = -z^t, \qquad I - z^* z = 0.$$

Going back to the previous realization of $\mathcal{D}$, a $2r$-dimensional complex subspace W belongs to the Shilov boundary of $\mathcal{D}$ if and only if

$$W \in S \Leftrightarrow B_{|W \times W} = 0, \qquad \mathrm{i}A_{|W\times W} = 0.$$

So W is totally isotropic for the full form $h = A+Bj$. Moreover, the space Wj is A-orthogonal to W and hence contained in W, so that W is a quaternionic subspace of V. The Shilov boundary Σ is realized as the space of all *quaternionic Lagrangian subspaces* (i.e. maximally h-isotropic subspaces) of V.

The Jordan algebra $J = \mathrm{Herm}(r, \mathbb{H})$ has a natural representation. Let us consider $F = (\mathbb{H}^r)_{\mathbb{R}}$ the real vector space of dimension $4r$ underlying the quaternionic vector space $\mathbb{H}^r$, and consider the Euclidean inner product defined by

$$(\xi, \eta) = \mathrm{Re}\langle \xi, \eta\rangle,$$

where $\langle \xi, \eta\rangle$ denotes the standard quaternionic inner product on $\mathbb{H}^r$. Then, for $x \in J$ and $\xi \in F$, let

$$\phi(x)\xi = x\xi.$$

This defines a Euclidean representation, and the index d equals $4r/r = 4$. As before, extend this representation to the complexification $\mathbb{J}$ and then use its restriction to the Shilov boundary in order to define the extended Maslov index.

The space $F \times F$ is the real vector space underlying the quaternionic vector space $\mathbb{V}$, and $\omega = \mathrm{Re}\, h$ is the corresponding (real) symplectic form on $F \times F$. Hence to a quaternionic Lagrangian space W in V is associated through the representation Φ the real underlying space $W_{\mathbb{R}}$, which is a Lagrangian for ω. Conversely, notice that if a Lagrangian of $V_{\mathbb{R}} \simeq F \times F$ for ω is indeed the real vector space underlying a quaternionic space W, then W is a Lagrangian subspace of V for h, as a consequence of formulae (19) and (20).

As a result, a Kashiwara's type definition of the Maslov index is available. For three quaternionic Lagrangians W_1, W_2, W_3 in V define Q on $W_1 \times W_2 \times W_3$ by

$$\begin{aligned} Q((\xi_1, \xi_2, \xi_3), (\eta_1, \eta_2, \eta_3)) = (&h(\xi_1, \eta_2) - h(\xi_2, \eta_1) + h(\xi_2, \eta_3) - h(\xi_3, \eta_2) \\ &+ h(\xi_3, \eta_1) - h(\xi_1, \eta_3)). \end{aligned} \tag{22}$$

This defines a quaternionic Hermitian form on $W_1 \times W_2 \times W_3$.

Theorem 5.1. *Let W_1, W_2, W_3 be three complex Lagrangians in V. The extended Maslov index is given by*

$$\iota(W_1, W_2, W_3) = \mathrm{sgn}(Q), \tag{23}$$

where sgn Q *is the signature of the quaternionic Hermitian form* Q *on* $W_1 \times W_2 \times W_3$ *defined by* (22).

The proof is almost the same as for the complex case, except that the signature over $\mathbb{H}$ of the Hermitian form Q is a *quarter* of the signature of the associated real quadratic form on $(W_1 \times W_2 \times W_3)_{\mathbb{R}}$. This is balanced by the fact that the index d equals 4 in this case.

6. The domains of type IV: elementary approach

A domain of type IV corresponds to the *Lorentzian* Euclidean Jordan algebra. Let $J = \mathcal{L}_p = \mathbb{R}^p = \mathbb{R} \oplus \mathbb{R}^{p-1}$ with the Jordan product

$$(x_1, x_2, \dots, x_p)(y_1, y_2, \dots, y_p) = (z_1, z_2, \dots, z_p)$$

with

$$z_1 = x_1 y_1 + x_2 y_2 + \cdots + x_p y_p, \qquad z_j = x_1 y_j + y_1 x_j \quad (2 \le j \le p)$$

and use the standard inner product on $\mathbb{R}^p$. The unit is $e = (1, 0, \dots, 0)$ and the associated cone is the future cone:

$$\Omega = \{(x_1, x_2, \dots, x_p) | x_1^2 - x_2^2 - \cdots - x_p^2 > 0, x_1 > 0\}.$$

In this section the classical geometric realization of the associated Hermitian symmetric domain $\mathcal{D}$ is recalled and its Shilov boundary S is determined. An elementary and explicit description of the open orbits in $S \times S \times S$ is given, as this does not seem to be available in the literature.

Let V be a real vector space, with a non-degenerate quadratic form S on V of signature $(p, 2)$, where $p \ge 2$. Let $\mathbb{V} = V \otimes_{\mathbb{R}} \mathbb{C}$ be its complexification, extend S to $\mathbb{V}$ as a complex bilinear symmetric form, still denoted by S, and on the other hand, let H be the Hermitian form on $\mathbb{V}$ defined by

$$H(x, y) = S(\bar{x}, y).$$

Consider the complex lines W in $\mathbb{V}$ which satisfy the following conditions:

$$S_{|W} = 0, \qquad H_{|W} \ll 0. \tag{24}$$

Fix once for all a basis $\{e_1, e_2, \dots, e_{p+1}, e_{p+2}\}$ such that

$$S(e_j, e_k) = 0 \quad \text{if} \quad j \ne k, \qquad S(e_j, e_j) = 1 \quad \text{if} \quad 1 \le j \le p,$$
$$S(e_j, e_j) = -1 \quad \text{if} \quad j = p+1 \quad \text{or} \quad p+2.$$

A line $W = \mathbb{C}\left(\sum_{j=1}^{p+2} z_j e_j\right)$ satisfies the conditions (24) if and only if

$$\begin{gathered} z_1^2 + \cdots + z_p^2 - z_{p+1}^2 - z_{p+2}^2 = 0, \\ |z_1|^2 + \cdots + |z_p|^2 - |z_{p+1}|^2 - |z_{p+2}|^2 < 0. \end{gathered} \tag{25}$$

These conditions imply $|z_{p+1}^2+z_{p+2}^2| < |z_{p+1}|^2+|z_{p+2}|^2$. Hence $z_{p+1} \neq 0$, $z_{p+2} \neq 0$, and $(z_{p+1}/z_{p+2})^2 \notin \mathbb{R}^+$, so $(z_{p+1}/z_{p+2}) \notin \mathbb{R}$. Hence $\mathfrak{J}(z_{p+1}/z_{p+2}) > 0$ or $\mathfrak{J}(z_{p+1}/z_{p+2}) < 0$. The space $\tilde{\mathcal{D}}$ of all complex lines satisfying (24) has two connected components, and let $\mathcal{D}$ be the component that corresponds to the condition:

$$\mathfrak{J}\left(\frac{z_{p+1}}{z_{p+2}}\right) > 0. \tag{26}$$

Let $W = \mathbb{C}\left(\sum_{j=1}^{p+2} z_j e_j\right)$ be in $\mathcal{D}$. Condition (26) implies that $z_{p+1} + \mathrm{i}z_{p+2} \neq 0$. Hence for $1 \leq j \leq p$, set

$$t_j = \frac{z_j}{z_{p+1} + \mathrm{i}z_{p+2}}.$$

Then observe that

$$\sum_{j=1}^{p} t_j^2 = \frac{z_{p+1}^2 + z_{p+2}^2}{(z_{p+1} + \mathrm{i}z_{p+2})^2} = \frac{z_{p+1} - \mathrm{i}z_{p+2}}{z_{p+1} + \mathrm{i}z_{p+2}}.$$

Moreover, notice that

$$\frac{|z_{p+1}|^2 + |z_{p+2}|^2}{|z_{p+1} + \mathrm{i}z_{p+2}|^2} = \frac{1}{2}\left(1 + \left|\frac{\mathrm{i}z_{p+1} + z_{p+2}}{z_{p+1} + \mathrm{i}z_{p+2}}\right|^2\right).$$

Hence the condition (25) now reads

$$\sum_{j=1}^{p} |t_j|^2 < \frac{1}{2}\left(1 + \left|\sum_{j=1} t_j^2\right|^2\right), \tag{27}$$

whereas the condition (26) amounts to $\left|\sum_{j=1}^{p} t_j^2\right| < 1$ (by Cayley transform from the upper half-plane into the unit disc), which thanks to the condition (27) is equivalent to the (seemingly) stronger condition:

$$\sum_{j=1}^{p} |t_j|^2 < 1. \tag{28}$$

Recall that $\mathbb{C}^p$ can be endowed with a structure of complex Jordan algebra with Jordan product defined by

$$(t_1, t_2, \ldots, t_p)(s_1, s_2, \ldots, s_p) = (u_1, u_2, \ldots, u_p),$$

where $u_1 = t_1 s_1 - t_2 s_2 - \cdots - t_p s_p$ and for $j \geq 2$, $u_j = t_1 s_j + t_j s_1$. The real form

$$J = \{(t_1, \mathrm{i}t_2, \ldots, \mathrm{i}t_p), t_j \in \mathbb{R}\}$$

is (isomorphic to) the Euclidean Jordan algebra of Lorentzian type of dimension p. The spectral norm in $\mathbb{V}$ is given by

$$N(t) = \left(\|\xi\|^2 + \|\eta\|^2 + 2\sqrt{\|\xi\|^2 + \|\eta\|^2 - (\xi,\eta)^2}\right)^{1/2},$$

where $t = (t_j)_{1\le j\le p}$, $t_j = \xi_j + \mathrm{i}\eta_j$, $\xi_j, \eta_j \in \mathbb{R}$ and $\xi = (\xi_1, \xi_2, \dots, \xi_p)$ and $\eta = (\eta_1, \eta_2, \dots, \eta_p)$ (see [8, Chapter X, Exercise 2]). The *Lie ball* is the domain defined by

$$\mathcal{B} = \{z \in \mathbb{C}^p | N(z) < 1\}.$$

The conditions (27) and (28) amount to say that W belongs to $\mathcal{D}$ if and only if $(t_1, t_2, \dots, t_p)$ belongs to $\mathcal{B}$. In fact observe that the condition (27) amounts to

$$\|\xi\|^2 + \|\eta\|^2 < \tfrac{1}{2}(1 + (\|\xi\|^2 - \|\eta\|^2)^2 + 4(\xi,\eta)^2)$$

or

$$\|\xi\|^2 + \|\eta\|^2 + \tfrac{1}{2}(1 + (\|\xi\|^2 - \|\eta\|^2)^2 - 4\|\xi\|^2\|\eta\|^2 + 4(\xi,\eta)^2),$$

which is equivalent to

$$4(\|\xi\|^2\|\eta\|^2 - (\xi,\eta)^2) < (1 + (\|\xi\|^2 + \|\eta\|^2)^2.$$

As $\|\xi\|^2 + \|\eta\|^2 < 1$, this is equivalent to

$$\|\xi\|^2 + \|\xi\|^2 + 2\sqrt{\|\xi\|^2 + \|\eta\|^2 - (\xi,\eta)^2} < 1, \tag{29}$$

which is the desired condition.

The Shilov boundary $S_{\mathcal{B}}$ of the domain $\mathcal{B}$ is the set

$$S_{\mathcal{B}} = \{\sigma \in \mathbb{C}^p | \sigma = \mathrm{e}^{-\mathrm{i}\theta}u,\ u \in \mathbb{R}^p,\ \|u\| = 1,\ \theta \in \mathbb{R}/2\pi\mathbb{Z}\}$$

(see [8]). As $\mathrm{e}^{\mathrm{i}\theta}u = \mathrm{e}^{-\mathrm{i}(\theta+\pi)}(-u)$, $S_{\mathcal{B}} \simeq S^{p-1} \times S^1/\mathbb{Z}_2$.

To find the Shilov boundary S of the domain $\mathcal{D}$, let $\sigma = \sum_{j=1}^p t_j e_j \in S_{\mathcal{B}}$, with $t_j = \mathrm{e}^{\mathrm{i}\theta}u_j$, $u_j \in \mathbb{R}$. Then:

$$\sum_{j=1}^p t_j^2 = \mathrm{e}^{-2\mathrm{i}\theta}\sum_{j=1}^p u_j^2 = \mathrm{e}^{-2\mathrm{i}\theta} = \frac{\cos\theta - \mathrm{i}\sin\theta}{\cos\theta + \mathrm{i}\sin\theta} \tag{30}$$

and hence a generator of the corresponding complex line in $\mathbb{V}$ is the element

$$(u,\ \cos\theta,\ \sin\theta) = \sum_{j=1}^p u_j e_j + \cos\theta e_{p+1} + \sin\theta e_{p+2}.$$

Theorem 6.1. *The correspondence*

$$\mathrm{e}^{-\mathrm{i}\theta}u \to \mathbb{C}(u + \sin\theta e_{p+1} + \cos\theta e_{p+2})$$

is a diffeomorphism of $S_{\mathcal{B}}$ onto S.

As a consequence, the Shilov boundary Σ of $\mathcal{D}$ is the set of all real *isotropic lines* in V.

The determinant corresponding to the Jordan algebra structure on $\mathbb{V}$ is given by

$$\Delta(z) = z_1^2 + z_2^2 + \cdots + z_p^2.$$

Two points of S, say $\zeta = \mathrm{e}^{-\mathrm{i}\theta}u$ and $\eta = \mathrm{e}^{-\mathrm{i}\phi}v$ are transverse if and only if

$$\zeta \top \eta \overset{\text{def}}{\Leftrightarrow} \Delta(\zeta - \eta) \neq 0 \Leftrightarrow \cos(\theta - \phi) \neq (u, v).$$

As

$$S((u, \cos\theta, \sin\theta), (v, \cos\phi, \sin\phi)) = (u|v) - \cos(\theta - \phi),$$

two isotropic lines W_1 and W_2 in V are transverse (this will be denoted by $W_1 \top W_2$) if and only if they are distinct and the plane they generate is not totally isotropic. It implies that the restriction of S to $W_1 \oplus W_2$ is non-degenerate, and of signature $(+, -)$.

The group $O(p, 2)$ has four connected components. To describe its neutral component $G = O(p, 2)^0$, first recall that for $g \in O(p, 2)$, $(\det g)^2 = 1$, so that $\det g = 1$ for $g \in G$ by connectedness. Then let $V = V_+^0 \in V_-^0$ be a fixed orthogonal decomposition such that the form is positive-definite on V_+^0 and negative-definite on V_-^0. Let p_- be the orthogonal projection on V_-^0. Then, for any subspace V_- of V such that the restriction of the form to V_- is negative-definite, the restriction of p_- to V_- is injective. Now let $g \in O(p, 2)$. Then $g(V_-^0)$ is such a subspace, hence the map

$$V_-^0 \xrightarrow{g} V_- \xrightarrow{p_-} V_-^0$$

is a linear isomorphism. Denote by $\delta_-(g)$ its determinant (which is not 0). Then

$$G = O(p, 2)^0 = \mathrm{SO}_0(p, 2) = \{g \in O(p, 2), \det g = 1, \delta_-(g) > 0\}.$$

Lemma 6.2. *Let* $w_1 = e_1 + e_{p+2}$, $w_2 = e_1 - e_{p+2}$. *Then*

$$S(w_1, w_1) = S(w_2, w_2) = 0, \quad S(w_1, w_2) = 2$$

and the stabilizer in G of the pair (w_1, w_2) *is isomorphic to* $\mathrm{SO}_0(p - 1, 1)$.

The first statement is obvious. For the second statement, let $g \in G$ which stabilizes both w_1 and w_2. It then stabilizes the plane $\Pi = \mathbb{R}e_1 \oplus \mathbb{R}e_{p+2}$, hence also its orthogonal $\Pi^\perp$, which is the space generated by $e_2, e_3, \ldots, e_p$ and e_{p+1}. The matrix of g is of the form

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & & & & 0 \\ \vdots & & \gamma & & \vdots \\ 0 & & & & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix},$$

where γ is a $p \times p$ matrix. The restriction of g to $\Pi^\perp$ stabilizes the restriction of S to $\Pi^\perp$, and hence $g_{|\Pi^\perp} \in O(\Pi^\perp)$. So $\gamma \in O(p - 1, 1)$. But $\operatorname{Det} g = \operatorname{Det}(\gamma) = 1$. From the orthogonal decomposition $V = (\mathbb{R}e_1 \oplus \cdots \oplus \mathbb{R}e_p) \oplus (\mathbb{R}e_{p+1} \oplus \mathbb{R}e_{p+2})$, it is easily seen that $\delta_-(g) > 0 \Leftrightarrow g_{2,2} = g_{p+1,p+1} > 0$, which shows that $\gamma \in \mathrm{SO}_0(p-1, 1)$.

Lemma 6.3. *G operates transitively on couples of vectors w_1, w_2 such that $S(w_1, w_1) = S(w_2, w_2) = 0$, $S(w_1, w_2) = 2$.*

Let (w_1, w_2) be any couple which satisfies the assumption. From Witt's theorem, there exists $g \in O(p, 2)$ such that $gw_1 = e_1 + e_{p+2}$, $gw_2 = e_1 - e_{p+2}$.

The element g is determined up to left multiplication by an element of $O(p, 2)$ which stabilizes both w_1 and w_2. The stabilizer in $O(p, 2)$ is isomorphic to $O(p-1, 1)$, hence has four connected components. But the matrices

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & \cdots & 0 & 0 \\ 0 & 0 & I_{p-2} & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & I_{p-2} & 0 & 0 \\ 0 & 0 & \cdots & -1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix},$$
$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & \cdots & 0 & 0 \\ 0 & 0 & I_{p-2} & 0 & 0 \\ 0 & 0 & \cdots & -1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

and the identity matrix I_{p+2} are in the stabilizer, and their corresponding restrictions give representatives of the four connected components of $O(p-1, 1)$. The lemma follows from these observations.

Denote as before by $S^3_\top$ the set of triples (W_1, W_2, W_3) such that $W_j \top W_k$ for $j \neq k$.

Theorem 6.4. *There are exactly three orbits in $S^3_\top$ under the action of the group G. Representatives (viewed as a triple of isotropic lines in V) of the orbits are given by*

- $(\mathbb{R}(e_1 + e_{p+2}), \mathbb{R}(e_1 - e_{p+2}), \mathbb{R}(e_1 + e_{p+1}))$;
- $(\mathbb{R}(e_1 + e_{p+2}), \mathbb{R}(e_1 - e_{p+2}), \mathbb{R}(e_1 - e_{p+1}))$;
- $(\mathbb{R}(e_1 + e_{p+2}), \mathbb{R}(e_1 - e_{p+2}), \mathbb{R}(e_2 - e_{p+2}))$.

Proof. Let $(W_1, W_2, W_3) \in S^3_\top$. As the restriction of S to $W_1 \oplus W_2$ is non-degenerate, choose $w_1 \in W_1$, $w_2 \in W_2$ such that

$$S(w_1, w_1) = S(w_2, w_2) = 0, \qquad S(w_1, w_2) = 2.$$

These conditions determine the couple (w_1, w_2) up to a real number $\lambda \neq 0$, in the sense that any other solution is of the form $\lambda\omega_1$, $(1/\lambda)w_2$. For a specific choice of w_1, there exists a unique $w_3 \in W_3$ such that $S(w_1, w_3) = 1$. Then $S(w_2, w_3) = \mu \neq 0$. Changing w_1 to λw_1

replaces w_2 by $(1/\lambda)w_2$ and w_3 by $(1/\lambda)w_3$, hence changes μ into $(1/\lambda)^2\mu$. It is always possible to assume that

$$S(w_1, w_3) = 1, \qquad S(w_2, w_3) = \varepsilon$$

with $\varepsilon = \pm 1$. Let $\Pi = \mathbb{R}w_1 \oplus \mathbb{R}w_2$, and $\Pi^\perp$ its orthogonal, so that $V = \Pi \oplus \Pi^\perp$. According to this decomposition, write $w_3 = p_3 + q_3$, with $p_3 \in \Pi$ and $q_3 \in \Pi^\perp$. Then it is easily verified that $p_3 = (1/2)(\varepsilon w_1 + w_2)$. Note further that, as $S(w_3, w_3) = 0$, $S(q_3, q_3) = -S(p_3, p_3) = -\varepsilon$.

Thanks to Lemma 6.2, it suffices to consider the case where $w_1 = e_1 + e_{p+2}$, $w_2 = e_1 - e_{p+2}$, so that $p_3 = e_1$ if $\varepsilon = 1$, and $p_3 = -e_{p+2}$ if $\varepsilon = -1$.

Recall that there are two orbits under $\mathrm{SO}_0(p-1, 1)$ in the set

$$\{x_2e_2 + \cdots + x_pe_p + x_{p+1}e_{p+1} | x_2^2 + \cdots + x_p^2 - x_{p+1}^2 < 0\}$$

with representative, respectively, $+e_{p+1}$ and $-e_{p+1}$, and one orbit in the set

$$\{x_2e_2 + \cdots + x_pe_p + x_{p+1}e_{p+1} | x_2^2 + \cdots + x_p^2 - x_{p+1}^2 > 0\}$$

with representative e_2. The case where $\varepsilon = 1$ gives rise to two orbits with representatives $q_3 = e_{p+1}$ and $q_3 = -e_{p+1}$ (and hence $w_3 = e_1 + e_{p+1}$, $w_3 = e_1 - e_{p+1}$) and the case where $\varepsilon = -1$ gives rise to one orbit, with representative $q_3 = e_2$ (and hence $w_3 = e_2 - e_{p+2}$). The theorem follows from these results. □

7. Isotropic lines in V and pure Lagrangians in the spinor space

The Lorentzian Jordan algebra $\mathcal{L}_p$ has non-trivial representations. The space E of such a representation is a Clifford module for the Clifford algebra constructed on the Euclidean space $\mathbb{R}^{p-1}$, and vice versa (see [3]). It is more involved that the previous cases. In order to describe the embedding of the Shilov boundary S (=the manifold of isotropic lines) of the domain $\mathcal{D}$ (=the Lie ball) into the corresponding Lagrangian manifold of $E \times E$ in geometric terms, it is more convenient, in agreement with the geometric presentation of the space $\mathcal{D}$ in Section 6 to start with a real vector space V of dimension $p + 2$ equipped with a quadratic form S of signature $(p, 2)$, to consider the associated *Clifford algebra* $C = C(V, S)$ and the associated space of spinors Σ. It turns out that Σ has a natural symplectic structure. For each isotropic line in V a certain Lagrangian subspace of Σ is constructed. This construction will be interpreted in Section 8 as an embedding of the Shilov boundary S into the Lagrangian manifold of Σ, associated to a holomorphic embedding of the Lie ball in the Siegel disc associated to the symplectic space Σ.

Recall that the Clifford algebra[4] $C = C(V, S)$ is the algebra generated over $\mathbb{R}$ by V with the relations $xy + yx = 2S(x, y)$. The space V itself will be regarded as a subspace in C. Let $^\wedge$ be the conjugation of C, i.e. the unique antiautomorphism of C such that $\hat{x} = -x$ for $x \in V$.

[4] A general reference for Clifford algebras and spinors is [9]. See also [6].

Let C^{ev} (resp. C^{odd}) be the even (resp. odd) part of $C(V, S)$. The structure of the algebra C^{ev} is well known. It is either simple or direct sum of two simple ideals. In any case, a simple ideal is isomorphic to a matrix algebra over $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. The algebra of $N \times N$ matrices with coefficients in $\mathbb{K}$ is denoted by $\mathrm{Mat}(N, \mathbb{K})$.

The description $\bmod 8$ of C^{ev} is given by the following list:

- Case $p \equiv 0\ (8)$: $C^{\text{ev}} \simeq \mathrm{Mat}(N, \mathbb{C})$, $N = 2^{p/2}$.
- Case $p \equiv 1\ (8)$: $C^{\text{ev}} \simeq \mathrm{Mat}(N, \mathbb{R})$, $N = 2^{(p+1)/2}$.
- Case $p \equiv 2\ (8)$: $C^{\text{ev}} \simeq \mathrm{Mat}(N, \mathbb{R}) \oplus \mathrm{Mat}(N, \mathbb{R})$, $N = 2^{p/2}$.
- Case $p \equiv 3\ (8)$: $C^{\text{ev}} \simeq \mathrm{Mat}(N, \mathbb{R})$, $N = 2^{(p+1)/2}$.
- Case $p \equiv 4\ (8)$: $C^{\text{ev}} \simeq \mathrm{Mat}(N, \mathbb{C})$, $N = 2^{p/2}$.
- Case $p \equiv 5\ (8)$: $C^{\text{ev}} \simeq \mathrm{Mat}(N, \mathbb{H})$, $N = 2^{(p-1)/2}$.
- Case $p \equiv 6\ (8)$: $C^{\text{ev}} \simeq \mathrm{Mat}(N, \mathbb{H}) \oplus \mathrm{Mat}(N, \mathbb{H})$, $N = 2^{(p-2)/2}$.
- Case $p \equiv 7\ (8)$: $C^{\text{ev}} \simeq \mathrm{Mat}(N, \mathbb{H})$, $N = 2^{(p-1)/2}$.

(see [9] for a proof of these facts).

In any case, denote by Σ a simple module under the action of C^{ev}. It is a (right) $\mathbb{K}$-module of dimension N, called "the" *space of spinors*.

Proposition 7.1. *There exists a* (*unique up to a non-zero real constant*) $\mathbb{K}$ *skew-Hermitian non-degenerate product h on Σ such that*

$$h(a\xi, \eta) = h(\xi, \hat{a}\eta)$$

for $\xi, \eta \in \Sigma$ and $a \in C^{\text{ev}}$. Moreover in the complex case ($\mathbb{K} = \mathbb{C}$), *the form h is split.*

For a proof, see [9].

A Lagrangian subspace is a $\mathbb{K}$ vector subspace of Σ of half dimension $(=N/2)$ which is totally isotropic for the form h. The spin group $\mathrm{Spin}(V, S)$ acts on Σ by the spinor representation, preserving the form h, and hence $\mathrm{Spin}(V, S)$ acts on the Lagrangian manifold of Σ. A specific orbit of this action will be described now.

Denote by $\mathcal{N}$ the cone of isotropic vectors in V

$$\mathcal{N} = \{x \in V | S(v, v) = 0\}$$

and by $\mathcal{N}^*$ the set of non-zero vectors of $\mathcal{N}$.

Lemma 7.2. *Let $x \in \mathcal{N}^*$. Set*

$$J_x = \{a \in C^{\text{ev}} | ax = 0\}.$$

Then $J_x = C^{\text{odd}} x$.

Fix an orthogonal decomposition of V as $V_+ \oplus V_-$, where the restriction of S to V_+ is positive-definite, whereas the restriction of S to V_- is negative-definite. For $x \in V$, set $x = x_+ + x_-$, with $x_+ \in V_+$ and $x_- \in V_-$. Hence, if $x \in \mathcal{N}$:

$$S(x_+, x_+) = -S(x_-, x_-), \qquad S(x_+, x_-) = 0.$$

If moreover $x \neq 0$, then $x_+, x_- \neq 0$, and as J_x does depend projectively on x, it is possible to assume that $S(x_+, x_+) = -S(x_-, x_-) = 1$. Introduce the following elements of C^{ev}:

$$a_x = \tfrac{1}{4}(x_+ - x_-)(x_+ + x_-), \qquad b_x = \tfrac{1}{4}(x_+ + x_-)(x_+ - x_-). \tag{31}$$

The following properties are easily verified:

$$\begin{aligned} &a_x^2 = a_x, \qquad b_x^2 = b_x, \qquad a_x b_x = b_x a_x = 0, \qquad a_x + b_x = 1,\\ &a_x x = 0, \qquad x a_x = x, \qquad x b_x = 0, \qquad b_x x = x. \end{aligned}$$

Let $J'_x = C^{\mathrm{odd}} x$. Clearly $J'_x \subset J_x$. Now let $c \in J_x$. Then

$$c = c(a_x + b_x) = c a_x + c x(\tfrac{1}{4}(x_+ - x_-)) = c a_x = c(\tfrac{1}{4}(x_+ - x_-))x \in J'_x.$$

Hence $J_x \subset J'_x$.

The proof also shows that

$$J_x = C^{\mathrm{odd}} x = C^{\mathrm{ev}} a_x.$$

Similarly let

$$H_x \stackrel{\mathrm{def}}{=} \{a \in C^{\mathrm{ev}} | xa = 0\} = x C^{\mathrm{odd}} = b_x C^{\mathrm{ev}}.$$

Lemma 7.3. *Let $x \in \mathcal{N}^*$, and let $\xi \in \Sigma$. Then*

$$\begin{aligned} &J_x \xi = 0 \Leftrightarrow a_x \xi = 0 \Leftrightarrow b_x \xi = \xi \Leftrightarrow \xi \in H_x \Sigma,\\ &H_x \xi = 0 \Leftrightarrow b_x \xi = 0 \Leftrightarrow a_x \xi = \xi \Leftrightarrow \xi \in J_x \Sigma. \end{aligned}$$

Proof. Let ξ be such that $J_x \xi = 0$. Then $a_x \xi = 0$, hence $b_x \xi = \xi$, so that $\xi \in H_x \Sigma$. Conversely, let $\xi \in H_x \Sigma$. This means that there exists $\eta \in \Sigma$, such that $\xi = b_x \eta$, and hence $a_x \xi = 0$, so that $J_x \xi = 0$. The proof of the second line is similar.

For $x \in \mathcal{N}^*$, define

$$\Sigma_x = \{\xi \in \Sigma | J_x \xi = 0\} = H_x \Sigma, \qquad \Sigma^x = \{\xi \in \Sigma | H_x \xi = 0\} = J_x \Sigma. \qquad \square \tag{32}$$

Lemma 7.4. *Let $x \in \mathcal{N}^*$. Then Σ_x and Σ^x are transverse Lagrangian subspaces of Σ.*

Proof. As the action of $\mathbb{K}$ commutes with the action of the Clifford algebra, Σ_x (resp. Σ^x) is a $\mathbb{K}$ vector subspace. Let $\xi, \eta \in \Sigma_x$. Then, as $\hat{a}_x = b_x$

$$h(\xi, \eta) = h(b_x \xi, \eta) = h(\xi, a_x \eta) = 0.$$

This shows that Σ_x is totally isotropic. A similar argument shows that Σ^x is also totally isotropic. For any $\xi \in \Sigma$, write $\xi = a_x \xi + b_x \xi$. The first term belongs to Σ^x and the second to Σ_x. Hence $\Sigma = \Sigma_x + \Sigma^x$, and as a totally isotropic subspace has a dimension less than half the dimension of the space, the dimensions of Σ_x and Σ^x are equal to $N/2$, and the conclusion follows. Observe moreover that $\Sigma^x = \Sigma_{x'}$ with $x' = x_+ - x_-$.

A Lagrangian subspace Λ in Σ which is of the form Σ_x (or equivalently Σ^x) for some $x \in \mathcal{N}^*$ is said to be *pure*. Denote by Λ^{pure} the subset of pure Lagrangians in the Lagrangian manifold of σ. □

Theorem 7.5. *The mapping*

$$x \mapsto \Sigma_x$$

is a bijective map from the projectivized space $\mathcal{N}^*/\mathbb{R}^*$ on Λ^{pure}, *which is equivariant with respect to the action of the spin group*.

Proof. Let us first prove the equivariance. Denote by $G = \text{Spin}(V, S)$ the spin group. Λ^{pure} is invariant under the action of G. Let $g \in G$. Let $x \in \mathcal{N}^*$, and let $g \cdot x = gxg^{-1}$. Then

$$c \in J_{g\cdot x} \Leftrightarrow cgxg^{-1} = 0 \Leftrightarrow cgx = 0 \Leftrightarrow cg \in J_x.$$

Hence $J_{g\cdot x} = J_x g^{-1}$. Now

$$\xi \in \Sigma_{g\cdot x} \Leftrightarrow J_{g\cdot x}\xi = 0 \Leftrightarrow J_x g^{-1}\xi = 0 \Leftrightarrow g^{-1}\xi \in \Sigma_x.$$

Hence $\Sigma_{gx} = g\Sigma_x$, which gives the equivariance property. The map is surjective by definition. The stabilizer P_x of the isotropic line $\mathbb{R}x$ is known to be a *maximal parabolic subgroup* of G, hence the stabilizer of Σ_x is either G or P_x. As the spinor representation is irreducible, it cannot be G. Hence it is equal to P_x. So if $\Sigma_x = \Sigma_y$, then $P_x = P_y$, and hence $\mathbb{R}x = \mathbb{R}y$. This finishes the proof. □

The last result of this section sheds a new light on the notion of transverse isotropic lines introduced in Section 6.

Proposition 7.6. *Let* $x, y \in \mathcal{N}^*$. *Then*

$$x \top y \Leftrightarrow \Sigma_x \cap \Sigma_y = \{0\}.$$

Proof. Let $x, y \in \mathcal{N}^*$ verifying $x \top y$. The restriction of S to the plane $\mathbb{R}x \oplus \mathbb{R}y$ is of signature $(+, -)$. Fix an orthogonal basis $\{f_+, f_-\}$ such that $S(f_+, f_+) = 1$, $S(f_-, f_-) = -1$. Let $x = \lambda f_+ + \mu f_-$. Then, as x is isotropic, $\lambda^2 = \mu^2$. By scaling of x and possibly changing f_- to $-f_-$, one can assume w.l.o.g. that $\lambda = \mu = 1$, hence $x = f_+ + f_-$. By a similar argument applied to y, one can assume that $y = f_+ - f_-$. The arguments used during the proof of Lemmas 7.2 and 7.3 imply that $\Sigma^y = \Sigma_x$ and hence $\Sigma_x \cap \Sigma_y = \{0\}$. Now suppose that on the contrary, x and y are not transverse. Then $S(x, y) = 0$, hence

$$0 \neq xy = -yx \in J_x \cap J_y$$

and so $\Sigma^x \cap \Sigma^y \neq \{0\}$. □

8. The case of type IV: the spinor approach

The embedding of the space of istropic lines of V into the space of pure Lagrangians as realized in Section 7 is associated to a representation of the Lorentzian Jordan algebra $\mathcal{L}_p$, and this will lead to a formula à la Kashiwara for the Maslov index of three isotropic lines in V (not necessarily tranverse).

In turn, such a representation is equivalent to a homomorphism:

$$\rho : \mathfrak{g} = \mathfrak{o}(V, S) \to \mathfrak{sp}(\Sigma),$$

which satisfies *Satake's condition* (H_2) (see [14, p. 84]). It turns out to be easier to start from such an embedding and then deduce the correspondence between the Shilov boundary S and the Lagrangian manifold. The first step is to interpret $\mathfrak{o}(V, S)$ as a Lie subalgebra of the Clifford algebra $C(V, S)$.

In fact, the Lie algebra $\mathfrak{g} = \mathfrak{o}(V, S)$ has a realization in C^{ev} as

$$\mathfrak{g} = \{x \in C^{\mathrm{ev}} | \hat{x} + x = 0, [x, V] \subset V\},$$

the Lie bracket being just the commutator in the Clifford algebra.

Fix an S-orthogonal basis $\{e_j\}_{1\le j\le p+2}$ of V, such that

$$S(e_j, e_j) = +1 \quad \text{for } 1 \le j \le p, \qquad S(e_j, e_j) = -1 \quad \text{for } j = p+1, p+2.$$

A concrete realization of $\mathfrak{g}$ is

$$\mathfrak{g} = \bigoplus_{1\le i<j\le p+2} \mathbb{R} e_i e_j.$$

Let $e_- = e_{p+1}e_{p+2}$. Observe that $e_-^{-1} = -e_- = \hat{e}_-$, and consider the inner automorphism of the Clifford algebra given by

$$\alpha : a \mapsto e_-^{-1} a e_-.$$

The automorphism α commutes with $^\wedge$ and preserves C^{ev}. It stabilizes V, and induces $+\mathrm{id}$ on the space generated by the e_j, $1 \le j \le p$, and $-\mathrm{id}$ on the space generated by e_{p+1}, e_{p+2}. The subspace $\mathfrak{g}$ is stable by the involution α and it induces a Cartan involution of $\mathfrak{g}$, hence a splitting $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ (see [14, Appendix]). The space $\mathfrak{p}$ is given by

$$\mathfrak{p} = \left(\bigoplus_{j=1}^{p} \mathbb{R} e_j e_{p+1}\right) \oplus \left(\bigoplus_{j=1}^{p} \mathbb{R} e_j e_{p+2}\right).$$

The element $H + 0 = (1/2)e_-$ is in the center of $\mathfrak{k}$ and satisfies

$$[H_0, e_j e_{p+1}] = e_j e_{p+2}, \qquad [H_0, e_j e_{p+2}] = -e_j e_{p+1},$$

hence induces a complex structure on $\mathfrak{p}$. The eigenspace of ad H_0 for the eigenvalue $+\mathrm{i}$ is denoted by $\mathfrak{p}_+$ and

$$\mathfrak{p}_+ = \left(\bigoplus_{j=1}^{p} \mathbb{C} e_j\right) \left(\frac{e_{p+1} - \mathrm{i} e_{p+2}}{2}\right) \simeq \mathbb{C}^p.$$

Let σ be the conjugation with respect to the compact real form $\mathfrak{k} \oplus \mathrm{i}\mathfrak{p}$. The involution σ maps $\mathfrak{p}_+$ into $\mathfrak{p}_-$. For $v \in \oplus_{j=1}^{p} \mathbb{C}e_j$:

$$e(v(\tfrac{1}{2}(e_{p+1} - \mathrm{i}e_{p+2}))) = (-\bar{v})(\tfrac{1}{2}(e_{p+1} + \mathrm{i}e_{p+2})).$$

Now let

$$e = e_1(\tfrac{1}{2}(e_{p+1} - \mathrm{i}e_{p+2})).$$

This is an element of $\mathfrak{p}_+$, and it satisfies

$$[e, \sigma(e)] = \mathrm{i}e_{p+1}e_{p+2} = 2\mathrm{i}H_0.$$

Moreover, let $v = z_1e_1 + z_2e_2 + \cdots + z_pe_p$. Then a straightforward computation shows that

$$-\tfrac{1}{2}\{e, v(\tfrac{1}{2}(e_{p+1} - \mathrm{i}e_{p+2})), e\} = v'(\tfrac{1}{2}(e_{p+1} - \mathrm{i}e_{p+2})),$$

where $v' = \bar{z}_1e_1 - \bar{z}_2e_2 - \cdots - \bar{z}_pe_p$. The corresponding real form (cf. Section 3) is

$$J = \{v(\tfrac{1}{2}(e_{p+1} - \mathrm{i}e_{p+2}))|v = x_1e_1 + \mathrm{i}(x_2e_2 + \cdots + x_pe_p), x_1, x_2, \ldots, x_p \in \mathbb{R}\}$$

isomorphic to $\mathbb{R}^p$, and the Jordan algebra structure coincides with the one presented in Section 6.

Recall form Section 7 that Σ is a spinor space for C^{ev}. If $a \in \mathfrak{g} \subset C^{\mathrm{ev}}$, then $\hat{a} = -a$, and hence

$$h(a\xi, \eta) + h(\xi, a\eta) = 0$$

for all $\xi, \eta \in \Sigma$. The form $A = \mathrm{Re}\, h$ is a symplectic form on Σ. By restriction of the action of C^{ev} on Σ, one obtains a homomorphism ρ of Lie algebras:

$$\rho : \mathfrak{o}(V, S) \to \mathfrak{sp}(\Sigma, A).$$

Our next goal is to show that this homomorphism satisfies Satake's condition (H_2) condition, namely:

$$\rho(H_0) = \tfrac{1}{2}J,$$

where $J \in \mathfrak{sp}(\Sigma)$ is a complex structure on Σ such that $A(\xi, J\eta)$ is a positive-definite form on Σ.

The Clifford algebra C is isomorphic (as vector space) to the *exterior algebra* $\Lambda(V)$. There is a canonical non-degenerate symmetric $\mathbb{R}$-bilinear form on $\Lambda(V)$, denoted by $\langle , \rangle$. It satisfies

$$\langle ab, c\rangle = \langle a, c\hat{b}\rangle = \langle b, \hat{a}c\rangle \tag{33}$$

for all a, b, c in C. In particular

$$\langle a, b\rangle = \langle 1, \hat{a}b\rangle = \langle 1, b\hat{a}\rangle. \tag{34}$$

The subspaces C^{ev} and C^{odd} are orthogonal for $\langle , \rangle$, and hence the restriction of $\langle , \rangle$ defines a non-degenerate symmetric bilinear form on C^{ev}. Similarly, in the non-simple case C^{ev}_{+} and C^{ev}_{-} are orthogonal. In fact, let a $a \in C^{\mathrm{ev}}_{+}$ and $b \in C^{\mathrm{ev}}_{-}$. Then

$$\langle a, b\rangle = \langle \lambda a, b\rangle = \langle a, \hat{\lambda} b\rangle = \langle a, \lambda b\rangle = -\langle a, b\rangle$$

so that $\langle a, b\rangle = 0$ for all $a \in C^{\mathrm{ev}}_{+}$ and $b \in a \in C^{\mathrm{ev}}_{-}$. So the restriction of the form $\langle , \rangle$ to $C^{\mathrm{ev}}_{\pm}$ is non-degenerate.

Proposition 8.1.

(i) *Assume* C^{ev} *is simple. For all* $a, b \in C^{\mathrm{ev}}$:

$$\langle a, b\rangle = \frac{1}{N}\,\mathrm{Tr}_{\mathbb{R}}\,\rho(a\hat{b}), \tag{35}$$

where $N = \dim \Sigma$.

(ii) *Assume* $p \equiv 2, 6 \bmod 8$. *For all* $a, b \in C^{\mathrm{ev}}_{\pm}$:

$$\langle a, b\rangle = \frac{2}{N}\mathrm{Tr}_{\mathbb{R}}\,\rho_{\pm}(a\hat{b}), \tag{35$'$}$$

where $N = \dim \Sigma_{\pm}$.

Proof. The notation for statement (ii) will be explicited during the proof. Assume C^{ev} is simple. Thanks to (34), it suffices to show that

$$\langle 1, a\rangle = \frac{1}{N}\mathrm{Tr}\,\rho(a)$$

for any $a \in C^{\mathrm{ev}}$. Let $\Psi(a) = (1/N)\mathrm{Tr}\,\rho(a)$. This defines a linear form Ψ on C^{ev}, which satisfies $\Psi(ab) = \Psi(ba)$. As $\langle , \rangle$ is non-degenerate on C^{ev}, there exists $c \in C^{\mathrm{ev}}$ such that $\Psi(a) = \langle a, c\rangle$ for all $a \in C^{\mathrm{ev}}$. Now

$$\langle ab, c\rangle = \langle ba, c\rangle \Rightarrow \langle a, c\hat{b}\rangle = \langle a, \hat{b}c\rangle$$

for any a, b, and hence $c\hat{b} = \hat{b}c$ for any $b \in C^{\mathrm{ev}}$, so that c must belong to the center of C^{ev}.

In the odd case, the center of C^{ev} is reduced to $\mathbb{R}1$, so c is a multiple of 1, and one checks that the normalizing constant $1/N$ is the proper one.

For the even case we need more notation. Let $\lambda = e_1 e_2 \cdots e_{p+1} e_{p+2}$ (*volume element*). Observe that λ up to a sign does not depend on the specific choice of the $(e_j)_{1\le j\le p+2}$. The center of C^{ev} is $\mathbb{R} \oplus \mathbb{R}\lambda$, so that $c = \alpha + \beta\lambda$, with $\alpha, \beta \in \mathbb{R}$. By setting $a = 1$:

$$\Phi(1) = \frac{1}{N}\mathrm{Tr}\,\rho(1) = 1 = \alpha\langle 1, 1\rangle + \beta\langle 1, \lambda\rangle = \alpha + 0.$$

Hence $\alpha = 1$. Set $a = \lambda$ to get

$$\Phi(\lambda) = \frac{1}{N}\mathrm{Tr}\,\rho(\lambda) = \beta\langle \lambda, \lambda\rangle.$$

As $\langle \lambda, \lambda\rangle = \pm 1$, it suffices to show that $\mathrm{Tr}\,\rho(\lambda) = 0$ to be able to conclude that $\beta = 0$.

If $p \equiv 0, 4 \bmod 8$, $\lambda^2 = -1$ and $\rho(\lambda)$ is a complex structure on Σ. Hence $\operatorname{Tr} \rho(\lambda) = 0$.

It remains to study the case where $p \equiv 2, 6 \bmod 8$. It corresponds to the proof of (ii). In this case, $\hat{\lambda} = \lambda$ and $\lambda^2 = 1$. Accordingly, the algebra C^{ev} splits as $C^{\mathrm{ev}}_+ \oplus C^{\mathrm{ev}}_-$. Now $C^{\mathrm{ev}}_+ \simeq \mathrm{Mat}(N, \mathbb{K})$ and let $\rho_\pm : C^{\mathrm{ev}}_+ \to \mathrm{End}_{\mathbb{K}}(\Sigma_\pm)$ be corresponding half-spin representations. Extend $\rho_\pm$ to C^{ev} by taking it equal to 0 on $C^{\mathrm{ev}}\mp$. Arguing as before, it is enough to compute $\operatorname{Tr}_{\mathbb{R}} \rho_+(\lambda)$. Now write

$$\lambda = \tfrac{1}{2}(1+\lambda) + \tfrac{1}{2}(-1+\lambda)$$

and observe that $1 + \lambda/2$ is the unit element in C^{ev}_+. Hence

$$\operatorname{Tr}_{\mathbb{R}} \rho_+(\lambda) = \operatorname{Tr} \rho_+(\tfrac{1}{2}(1+\lambda)) = \dim \Sigma_+ = N.$$

Hence, for $a \in C^{\mathrm{ev}}_+$:

$$\frac{1}{N}\operatorname{Tr} \rho_+(a) = \langle a, 1\rangle + \langle a, \lambda\rangle = 2\langle a, 1\rangle.$$

As above, this implies (35'). The proof for ρ_- is similar.

Let $e_- = e_{p+1}e_{p+2}$. Observe that $e_-^{-1} = -e_- = \hat{e}_-$, and consider the automorphism

$$\alpha : a \mapsto e_-^{-1} a e_-.$$

The automorphism α commutes with $^\wedge$ and preserves C^{ev}. Moreover the quadratic form

$$a \mapsto \langle \alpha(a), a\rangle$$

is positive-definite on C^{ev} (see [14]).

Let $J = \rho(e_-)$. Then

$$J^2 = -1, \qquad h(J\xi, \eta) = -h(\xi, J\eta)$$

for all $\xi, \eta \in \Sigma$. As a consequence, $(\xi, \eta) = h(J\xi, \eta)$ is a $\mathbb{K}$-Hermitian non-degenerate form on Σ. □

Proposition 8.2. *The form* $(\,,\,)$ *defined on* Σ *by*

$$(\xi, \eta) := h(J\xi, \eta)$$

is positive-definite or negative-definite.

Proof. The proof will be given in the case where C^{ev} is simple, but the proof for the non-simple case is almost identical. Let T in $\mathrm{End}_{\mathbb{K}}(\Sigma)$, and denote by $\hat{T}$ its adjoint with respect to h, and by T^* its adjoint with respect to $(\,,\,)$. Then

$$T^* = -J\hat{T}J.$$

Hence $\rho(\alpha(a))^\wedge = -J\widehat{\rho(a)}J = \rho(a)^*$ for any $a \in C^{\mathrm{ev}}$, and hence $\operatorname{Tr}_{\mathbb{R}}(\rho(a)^*\rho(a))$ is positive-definite on C^{ev}. It implies that $\operatorname{Tr}_{\mathbb{R}}(T^*T)$ is a positive-definite form on $\mathrm{End}_{\mathbb{K}}(\Sigma)$. Now let $\xi \in \Sigma$. Let T_ξ be the operator on Σ defined by

$$T_\xi(\eta) = \xi(\xi, \eta).$$

Then $T_\xi^* = T_\xi$, and $\mathrm{Tr}_{\mathbb{R}}(T_\xi^* T_\xi) = \dim \mathbb{K}(\xi, \xi)^2$. If $\xi \neq 0$, then $(\xi, \xi)^2$ can never be 0. The real quadratic form (ξ, ξ) has no non-trivial isotropic vector, hence is either positive-definite or negative-definite.

By changing h to $-h$ if necessary, the form $(\cdot, \cdot)$ can be assumed to be positive-definite. Hence the homomorphism $\rho : \mathfrak{o}(V, S) \to \mathfrak{sp}(\Sigma)$ satisfies the (H_2) condition. So the map ρ maps the Shilov boundary of $\mathcal{D}$ (the Lie ball) into the corresponding Shilov boundary of the Siegel disc associated to Σ.

In the present realization of $\mathfrak{p}_+$, the Shilov boundary of $\mathcal{D}$ is (cf. Section 6)

$$S = \{\mathrm{e}^{\mathrm{i}\theta}\mu(\tfrac{1}{2}(e_{p+1} - \mathrm{i}e_{p+2}))\},$$

where $u \in V_+$, $\|u\| = 1$, $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, and $(-u, \theta + \pi)$ corresponds to the same point as (u, θ).

Let $\Sigma_{\mathbb{C}}$ be the complexification of Σ, and let $\mathbb{V}_-^0 = \{\xi \in \Sigma_{\mathbb{C}} | J\xi = -\mathrm{i}\xi\}$. Let $\mathbb{R}x$ be an isotropic vector in V. As before, normalize x so that

$$x = u + \cos\theta e_{p+1} + \sin\theta e_{p+2}$$

with $u \in \oplus_{j=1}^{p}\mathbb{R}e_j$, $\|u\| = 1$. In the present realization of the Shilov boundary, this corresponds to the point $\mathrm{e}^{\mathrm{i}\theta}u(e_{p+1} - \mathrm{i}e_{p+2})$. Now associate to this the point $\rho(\mathrm{e}^{\mathrm{i}\theta}u(e_{p+1} - \mathrm{i}e_{p+2}))$ and take the corresponding Lagrangian space of Σ:

$$W = W_x = \{\xi + \mathrm{e}^{\mathrm{i}\theta}u(e_{p+1} - \mathrm{i}e_{p+2})\xi | \xi \in \mathbb{V}_-^0\} \cap \Sigma. \qquad \square \tag{36}$$

Theorem 8.3. *The space W_x defined by (36) is a pure Lagrangian space.*

Proof. The space $\{\xi + \mathrm{e}^{\mathrm{i}\theta}u(e_{p+1} - \mathrm{i}e_{p+2})\xi | \xi \in \mathbb{V}_-^0\}$ is (by construction or by direct checking) a complex Lagrangian space in $\Sigma_{\mathbb{C}}$, which is stable by the conjugation with respect to Σ. Hence

$$W_x = \{\mathrm{Re}(\xi + \mathrm{e}^{\mathrm{i}\theta}u(e_{p+1} - \mathrm{i}e_{p+1}))\xi | \xi \in \mathbb{V}_-^0\}.$$

An element of $\mathbb{V}_-^0$ is of the form $\xi = (1 + \mathrm{i}e_{p+1}e_{p+2})\eta$ with η a real spinor ($\eta \in \Sigma$). Now

$$(\tfrac{1}{2}(e_{p+1} - \mathrm{i}e_{p+2}))(1 + e_{p+1}e_{p+2})\eta = (e_{p+1} - \mathrm{i}e_{p+2})\eta$$

so that

$$\mathrm{Re}(\xi + \mathrm{e}^{\mathrm{i}\theta}u(e_{p+1} - \mathrm{i}e_{p+2})\xi) = (1 + u(\cos\theta e_{p+1} + \sin\theta e_{p+2}))\eta$$

and hence, by comparison with (31) and (32)

$$W_x = (1 + u(\cos\theta e_{p+1} + \sin\theta e_{p+2}))\Sigma = a_x\Sigma = \Sigma^x$$

showing that W_x is a pure Lagrangian space. $\square$

Theorem 8.4. *The extended Maslov index for a triple of isotropic lines $(\mathbb{R}x_1, \mathbb{R}x_2, \mathbb{R}x_3)$ is given by*

$$\iota(\mathbb{R}x_1, \mathbb{R}x_2, \mathbb{R}x_3) = \frac{2}{N\dim(\mathbb{K})}\iota_\Sigma(W_{x_1}, W_{x_2}, W_{x_3}),$$

where ι_Σ is the (classical) Maslov index for triples of Lagrangians in the symplectic space $(\Sigma, \operatorname{Re} h)$.

This is formula (14) applied to our case (the rank of $\mathcal{L}_p$ is 2). Needless to say, the Lagrangians W_{x_i} are (right) $\mathbb{K}$-subspaces, and so this formula can also be written as

$$\iota(\mathbb{R}x_1, \mathbb{R}x_2, \mathbb{R}x_3) = \frac{2}{N}\tilde{\iota}_\Sigma(W_{x_1}, W_{x_2}, W_{x_3}),$$

where now $\tilde{\iota}$ means the Maslov index for Lagrangians of the $\mathbb{K}$ vector space (Σ, h)) (as defined in Sections 4 and 5).

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